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Dynamic Expectations Formation and U.S. Monetary Policy Regime Change

Xin Wei*

Abstract

This essay studies the fundamental causes of the monetary policy regime switches within rational expectations models. I introduce a threshold-switching monetary policy process into the model that links the policy stance to the fundamental shocks by an autoregressive regime strength index. It creates an expectations feedback mechanism between private agents' policy forecasts and future policy regime outcomes. As demonstrated in a novel threshold-switching Fisherian model, well management of the private sector's expectations of policy regime change can have the same effect as actually switching the regime. Contrastingly, failure to do so leads to unfavorable outcomes of policy intention. Then, I embed the new mechanism into a New Keynesian model. Along the way, I also develop an efficient non-simulation based threshold-switching Kalman filter, in conjunction with a solution method that accounts for the endogeneity of switching regimes, to estimate the nonlinear New Keynesian model. My key empirical findings are threefold. First, non-policy shocks have been instrumental in driving U.S. monetary policy regime changes during the post-World War II period. Most notably, markup shock explains 65.6% of variations in the policy regimes. Second, absent from markup shocks, eight of the eleven less aggressive regimes would not have happened during this history. Finally, I conclude that linking the private sector's dynamic expectations formation and the Fed's dilemma of the dual mandate in the presence of adverse supply shocks is a promising path towards providing micro-foundations for monetary policy regime shifts.

JEL classification: C11, C32, E31, E52

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1 Introduction

The past two decades have seen an absolute explosion of literature studying the time variations in the U.S. monetary policy. Early contributors in this field include Taylor (1999), Clarida, Galí and Gertler (2000), Cogley and Sargent (2001), Stock and Watson (2002), Lubik and Schorfheide (2004), Primiceri (2005), Boivin and Giannoni (2006) among others.¹ One important line of subsequent work recognizes that a policy change is not once-and-for-all and hence introduces Markov-switching processes to characterize the recurrence and stochastic nature of monetary policy shifts [Sims and Zha (2006), Davig and Leeper (2007), Bianchi (2012), Davig and Doh (2014), Bianchi and Melosi (2018), etc.]. However, while there has been an extensive discussion, some fundamental questions have not been touched upon: why have monetary policy regimes shifted, and what are the driving forces? Nothing can be said about it under the common exogenous Markov switching framework as far as I can tell. In this paper, I hope to investigate this by offering a model embedding a Taylor-type monetary policy process with threshold switching. These switches are modeled as the central bank's purposeful response to economic conditions and produce profound implications on economic equilibrium through agents' dynamic expectations formation of future policy change.

Figure 1 nicely motivates my research question. In panel A, the solid blue curve presents the actual federal funds rate, and the dashed red curve indicates the policy rate implied by the Taylor-rule. The U.S. post-World War II history demonstrates that the Taylor rule nicely outlines the trend of the Federal Reserve's monetary policy behavior. Panel B displays the discrepancies between the actual policy rate and the Taylor rule implied rate, which exhibits sizable magnitude and substantial persistence. The nature of these discrepancies suggests that they can hardly be folded into an additive, unanticipated shock component to the Taylor rule. It vividly reflects an essential limitation of the Taylor rule pointed out in literature: the parameters characterizing the responsiveness of policy rate to inflation and output gaps are constant.

Besides, the repeated tightening and easing episodes of monetary policy, as shown in Panel B, demonstrate that policy change is reversible. As long as there is no institutional reform that mechanically binds the monetary authority's policy choices, a rational agent would never rule out future policy reversal as a possibility in the spirit of Lucas (1976). For this reason, the literature has adopted Markov-switching in responsiveness coefficients of Taylor rule to characterize the repetitive pattern in policy changes. As a critical implication of the Markov-switching framework, Leeper and Zha (2003) separate the response to exogenous shocks into two components. They call one part

¹ The standard strategy in these papers has been to estimate models incorporating Taylor-type [Taylor (1993)] monetary policy rule over subsamples. The models commonly used include single-equation monetary policy reaction function regression, structural vector autoregression, and New Keynesian DSGE models.

² The evidence suggested in Figure 1 is robust to minor changes in the policy rule's functional form. For instance, the evidence remains true if using the original non-inertial Taylor rule specified in Taylor (1993, 1999) or modifying the degree of policy inertia. See, for example, Clarida et al. (2000) for estimates of the degree of policy inertia.

Figure 1. Federal Funds Rate: Actual versus Implied by Taylor Rule

Note: In the upper panel, the dashed red line indicates the Taylor rule-implied federal funds rate, and the solid blue line shows the effective federal funds rate. I specify the Taylor rule as an inertial policy rule based on Taylor (1999), i.e., $i_t = 0.85i_{t-1} + 0.15[4 + 1.5(\pi_t - 2) + y_t]$. i_t , π_t , and y_t denote the short-term nominal rate, inflation rate from four quarters ago, and the output gap. 0.85 is the degree of monetary policy inertia.² Data sample spans from 1954:Q3 - 2007:Q4 to exclude the zero lower bound situation in recent years. The Taylor rule is initialized at 1954:Q2 with the non-inertial Taylor rule implied rate. Both curved lines are plotted using quarterly data downloaded from FRED. The bottom panel plots the difference calculated as the actual rate minus Taylor rule implied rate. The shaded bars denote recessions defined by the National Bureau of Economic Research (NBER).

as direct effects that correspond to the economy with a fixed policy regime. The other component expectations-formation effects stem from the private sector reshaping its beliefs about future policy regime evolution in the face of new information. But, restricted by the exogenous Markov-switching framework, the agents in their model consider the true probability distribution across policy regimes as constant over time, or static. As a result, though very useful for evaluating modest policy interventions, their framework is not suitable for analyzing dynamic interplays between monetary authority and private agents, especially when the policy is immodest.³ In particular, if policymakers want to consider the private sector's dynamic expectations about future policy change in their current decisions, policy advisors would need a new framework.

The first contribution of this research is to detect the fundamental sources of U.S. monetary

³ Leeper and Zha (2003) defines modest policy interventions as intra-regime policy changes that only modestly alter agents' inference on the prevailing policy regime and thus do not induce significant expectations-formation effects.

policy regime shifts within a novel regime-switching DSGE model. I consider a threshold-switching monetary policy process with expectations feedback, where policy regimes are determined by the sign of a latent autoregressive regime strength variable. Different regimes represent different degrees of responsiveness in adjusting the policy rate to last period's policy rate, inflation and output deviations from their desired levels, and setting the inflation targets. A more responsive reaction to the inflation gap identifies the more aggressive regime, which is associated with a positive regime strength variable. The expectations feedback arises from the form of endogenous regime switches with respect to exogenous shocks. First, the Fed systematically adjusts the policy rate to the past policy rate, inflation, and output gaps, as well as choosing inflation targets. The economy's fundamental shocks, including policy and non-policy shocks, represent new non-target information to the monetary authority. These structural shocks will generate cumulative impacts on the future policy regime index that ultimately triggers the Fed to switch its policy regime. This channel is novel in the literature. Second, the policy behavior stochastically switches between the two regimes. The possible future regime changes affect private agents' expectations of future endogenous variables, which, in turn, alters the observed equilibrium values of current economic variables and feeds back to today's policy decisions. The expectations feedback through future regime changes capture the nonlinear features in the data.

The estimation result suggests that markup shocks have been a predominant factor in determining the U.S. monetary policy regime change. It contributes 65.6% of variations in the monetary policy regime strength index during the post-World War II period. Absent from adverse markup shocks, eight of the eleven identified less aggressive regimes would not have happened during this history. Out of the remaining three, the two most pronounced less aggressive regimes' regime strength during the 1970s is enormously weaker. The evidence from this new model economy supports the widespread knowledge that the two oil crises in 1973 and 1979 lead to the excessive easing in monetary policy stance. Then, I put forward a suggestive mechanism for the adverse markup shock's impact on the monetary policy shifts. The mechanism aims to describe a plausible strategy of the Fed confronting the rising inflation and declining output, resulting from adverse supply shocks, with the dual mandate. On the one hand, the Fed switches to the less aggressive regime to promote economic growth. On the other hand, it strategically maneuvers the private sector's dynamic expectations on future policy regimes to manage inflation expectations. Well-managed inflation expectation results in lower inflation through the expectational channel.

The second contribution is to gauge the agents' dynamic expectations formation via a parsimonious Fisherian model. From the methodological perspective, this paper invents a new analytical approach for policymakers to evaluate the private sector's expectations formation in a Fisherian model embedding a threshold-switching Taylor rule. The analytical solution to the model extends the work in Davig and Leeper (2007) and embeds their exogenous-switching results as a limiting case. To my best knowledge, analytical solutions to rational expectation models with threshold

switching did not exist in literature. The new derivation employs the guess-and-verify approach, where a correct guess of the nonlinear solution form is the key. The availability of an analytical solution makes transparent new features arising from the threshold-type endogenous switching. Researchers can potentially apply it to other parsimonious rational expectations models with endogenous switching.

Based on the analytical solution, using a calibrated Fisherian model, I demonstrated that dynamic expectational effects could be strikingly significant, sometimes entirely offsetting or even dominating those direct effects. Therefore, policymaker's management of the private sector's expectations can have the same level of effectiveness as actual policy stance changes. For instance, the monetary authority's promise to fight inflation whenever a significant expansionary shock hits the economy could induce the private sector to behave in favor of the central bank. As a result, inflation declines even though the monetary policy does not switch to fight inflation aggressively. On the other hand, a not-so-well managed expectations may cause to reverse the intended policy outcomes. Suppose an expansionary shock hits the economy. If the monetary authority mechanically switches to the more aggressive regime attempting to control inflation, without careful consideration of the private sector's expectations formation, agents' strong expectations of reverting to the less aggressive regime could amplify the inflationary effect of the underlying shock. The resulting inflation would be as high as the case that policy stays in the less aggressive regime. These results highlight the quantitative and qualitative importance of taking dynamic expectations formation into account in actual policymaking and lend a theoretical foundation to the inflation management mechanism I mentioned earlier.

Along the way, this paper also makes an econometric contribution. I develop a threshold-switching Kalman filter to estimate the new threshold-switching DSGE model. The new filter extends Kim (1994)'s filter for exogenous-switching state-space models to a threshold-type endogenous switching environment. Compared to simulation-based filters,⁴ my filter dramatically simplifies the computation by exploiting properties of the threshold-switching state-space models' underlying data-generating process. Notice that Kim's filter's essence is to collapse mixture normal distributions arising from exogenous-switching into normal distributions. The threshold-switching Kalman filter further collapses the mixture truncated normal distributions resulting from the threshold-switching as normal distributions. Compared to a textbook bootstrap particle filter, the numerical simulation shows that the new filter is computationally much more efficient without sacrificing any accuracy.

In the next section, I solve a threshold-switching Fisherian model analytically and demonstrate the importance of dynamic expectations formation. Section 3 of the paper introduces the threshold-switching Taylor rule into a New Keynesian model and computes a first-order perturbation solution.

⁴ For example, Kang (2014) develops a simulation-based filter for nonlinear state-space models embedding a reduced form endogenous switching process studied by Kim, Piger and Startz (2008).

Section 4 presents the estimation strategy, reports empirical analysis, and discusses theoretical implications. Section 5 concludes the essay.

2 Threshold-Switching Monetary Policy in a Fisherian Model

This section considers a simple inflation determination model, i.e., the Fisherian model, featuring a new threshold-switching monetary policy process. I derive an analytical solution for this parsimonious model and show that it reduces to the solutions for the exogenous-switching and the fixed-regime models as limiting cases. These solutions make transparent the rich features arise from the new model. Specifically, I demonstrate how this model modifies the expectations formation effects and macroeconomic dynamics.

2.1 The Model

This model of inflation determination links a threshold-type switching Taylor rule with a Fisher equation. The Fisher equation is commonly written as

$$i_t = r_t + \mathbf{E}_t \pi_{t+1}. \quad (1)$$

Here, $\mathbf{E}_t \pi_{t+1}$ is the expected inflation conditional on the information set that will be defined explicitly below in (8). i_t and r_t denote the short-term nominal interest rate and the (ex-ante) real interest rate. This equation is a widely accepted approximation of the exact relation of these variables, namely, $(1 + i_t) = (1 + r_t)(1 + \mathbf{E}_t \pi_{t+1})$. For simplicity and the purpose of exposition, r_t fluctuates around the natural rate of interest, r^* , with respect to a real rate shock

$$r_t = r^* + \sigma_r \varepsilon_t^r, \quad \varepsilon_t^r \sim \mathcal{N}(0, 1), \quad (2)$$

where real rate innovation ε_t^r is standard normal, and r^* represents the real rate consistent with stable inflation absent any shocks. There is no persistence in the real rate process. The simplicity brings clarity and allows us to focus on the contemporaneous effect arising from threshold switching.

The monetary policy responds systematically to the inflation gap, with inflation target normalized to zero. The level of responsiveness switches between two regimes. That is,

$$i_t = \alpha(s_t) \pi_t \quad (3)$$

with $\alpha_0 < \alpha_1$, where $\alpha_0 = \alpha(s_t = 0)$ and $\alpha_1 = \alpha(s_t = 1)$. The regime is determined by the sign of a policy regime strength variable and is written as an indicator function

$$s_t = \mathbb{1}\{s_t^* \geq 0\}, \quad (4)$$

where the autoregressive policy regime strength index is

$$s_{t+1}^* = \delta(1 - \phi) + \phi s_t^* + w_{t+1}, \quad (5)$$

with $|\phi| < 1$ and δ reflecting the unconditional mean of regime strength. The innovation w_{t+1} to the regime strength index s_{t+1}^* is allowed to interact with the real rate innovation ε_t^r in the previous period and I specify their joint distribution as ⁵

$$\begin{pmatrix} \varepsilon_t^r \\ w_{t+1} \end{pmatrix} \sim \mathcal{N} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right). \quad (6)$$

Here, the correlation coefficient ρ quantifies the influence from ε_t^r to w_{t+1} through the feedback channel, and I restrict $|\rho| < 1$.

It follows from (5) and (6) that ⁶

$$s_{t+1}^* = (1 - \phi)\delta + \phi s_t^* + \rho \varepsilon_t^r + \sqrt{1 - \rho^2} v_{t+1}, \quad v_{t+1} \sim \mathcal{N}(0, 1). \quad (7)$$

Equation (7) highlights the channel from current real rate shock ε_t^r to next period regime selection s_{t+1}^* through regime strength index s_{t+1}^* . The endogeneity parameter ρ measures the strength of the feedback channel. When $\rho = 0$, monetary policy regime shift is completely based on the non-fundamental regime index shock v_{t+1} and the model collapses to the exogenous-switching case. When $|\rho| = 1$, regime at $t + 1$ becomes predetermined at t . It indicates that policymakers do not consider any non-fundamental information beyond that embedded in the history of real rate shocks when selecting policy regimes. This case corresponds to the setup considered in Davig, Leeper, Clarida and Lindé (2006). The authors assume regime at $t + 1$ to be predetermined by inflation π_t crossing a threshold level $\bar{\pi}$. Using notations in my paper, their specification writes as $s_{t+1} = \mathbb{1}\{\pi_t - \bar{\pi} \geq 0\}$. Notice that, in a simple Fisherian model, π_t is nothing but a combination

⁵ The regime-switching specification follows Chib and Dueker (2004) but with slight modification. For convenience, I replace their contemporaneous correlation of ε_t^r and w_t to $\mathbf{E}(\varepsilon_t^r w_{t+1}) = \rho$. It highlights the mechanism that private agents use new information embedded in ε_t^r to forecast the next period's regime strength s_{t+1}^* through w_{t+1} , which immediately leads to $\mathbf{E}(\varepsilon_t^r s_{t+1}^*) = \rho$. Notice that this modification is non-essential from an econometric point of view. Under the original specification of Chib and Dueker (2004), we would have $\mathbf{E}(\varepsilon_t^r s_{t+1}^*) = \phi\rho$, which amounts to a reparameterization. Nevertheless, the new parameterization used here makes the analytics more transparent. Another difference is that the authors' original model is introduced in the reduced form regression environment, while my paper extends it to the structural DSGE model. It creates a nontrivial challenge in the estimation, which I will address in Appendix C.

⁶ Projecting w_{t+1} onto the vector of ε_t^r yields

$$w_{t+1} = \frac{\mathbf{Cov}(w_{t+1}, \varepsilon_t^r)}{\mathbf{Var}(\varepsilon_t^r)} \varepsilon_t^r + \sqrt{1 - \rho^2} v_{t+1} = \rho \varepsilon_t^r + \sqrt{1 - \rho^2} v_{t+1}.$$

Here, $v_{t+1} \sim \mathcal{N}(0, 1)$, $\rho \varepsilon_t^r$ represents the least squares linear predictor of w_{t+1} , and $\sqrt{1 - \rho^2} v_{t+1}$ is the prediction error. One potential extension of current work is to include other endogenous variables and predetermined covariates to improve the least squares prediction.

of current and past real rate shocks. My setup takes an empirical perspective and will extract the latent regime strength index from data rather than presuming it to be past inflation.

Iterating s_{t+1}^* in (7) backwards yields

$$s_{t+1}^* = (1 - \phi^{t+1})\delta + \sum_{n=0}^t \phi^n (\rho \varepsilon_{t-n}^r + \sqrt{1 - \rho^2} v_{t+1-n}).$$

The autoregressive coefficient ϕ captures policy persistence since it measures the extent historical shocks would affect policy regime selection. One extreme case with $\phi = 0$ would mean that such an influence on policy regime selection only lasts one period. For the other extreme case with $\phi = 1$, policymakers put equal weights on information over the entire history. In the latter case, regime evolution has the highest persistence.

I assume the monetary authority and private agents observe the history of endogenous variables $(i_t), (\pi_t)$, exogenous variables and fundamental shocks $(r_t), (\varepsilon_t^r)$, and policy regime (s_t) , but none of the regime strength index or the non-fundamental shock $(s_t^*), (v_t)$.⁷ Then, agents' inflation expectation is

$$\mathbf{E}_t \pi_{t+1} = \mathbf{E}(\pi_{t+1} \mid \mathcal{G}_t), \quad \mathcal{G}_t = \{i_n, \pi_n, r_n, \varepsilon_n^r, s_n\}_{n=0}^t. \quad (8)$$

2.2 Analytical Solutions

I solve the expectational nonlinear difference equations consisting of (1)~(6) using the guess and verify method. It amounts to finding a fixed point in agents' decision rules. Davig and Leeper (2007) derive a minimum state variable solution as $\pi_t = a(s_t)r_t$, with a pair of constants $a(s_t)$, for a model with exogenous switching policy rule. Motivated by their solution, I start with the following guess for the solution form

$$\pi_t = a(s_t, p_{s_t,0}(\varepsilon_t^r))r_t^* + b(s_t)\varepsilon_t^r, \quad (9)$$

where $a(\cdot)$ and $b(\cdot)$ are two pairs of undetermined coefficients, $p_{s_t,0}(\varepsilon_t^r)$ is the regime- s_t to regime-0 time-varying transition probability (TVTP). It is worth noting that the regime- s_t to regime-1 TVTP, $p_{s_t,1}(\varepsilon_t^r)$, can be simply computed as $1 - p_{s_t,0}(\varepsilon_t^r)$. The TVTPs will be derived in Appendix A.2. This solution form presents the equilibrium characteristics that current inflation responds

⁷ This assumption of information structure is critical. Extensions of my work can be made by modifying this information structure. First, one may allow both the central bank and private agents to observe regime strength (s_t^*) , and hence (v_t) . It is more challenging to derive analytical solutions in this case. Specifically, the minimum state variables would be (ε_t^r, s_t^*) contrasting (ε_t^r, s_t) under the current assumption. However, it would be interesting to see how expectations formation effects and macroeconomic dynamics vary when agents and policymakers can observe the additional information on regime strength. Second, one may assume that the regime's strength remains unobservable to private agents but is known to the central bank. It mimics reality better but requires modeling the agents' learning dynamics of the regime strength index.

to the minimum state variables (ε_t^r, s_t) and that real rate shock induces a nonlinear effect on inflation through the transition probabilities of policy regime. Combining equations (1) and (3) and substituting the real rate process yield a regime-dependent equilibrium relation

$$\alpha(s_t)\pi_t = \mathbf{E}_t \pi_{t+1} + r^* + \sigma_r \varepsilon_t^r.$$

Let $\pi_{it} = \pi(s_t = i)$ for $i = 0, 1$. The solutions are ⁸

$$\pi_{0t} = \underbrace{\frac{1 - \omega_0(\varepsilon_t^r)}{\alpha_0 - 1 + \mathbf{E}[\omega_0(\varepsilon^r)]}}_{a_0^N} r^* + \underbrace{\frac{\sigma_r}{\alpha_0}}_{b_0} \varepsilon_t^r, \quad (10)$$

and

$$\pi_{1t} = \underbrace{\frac{1 - \omega_1(\varepsilon_t^r)}{\alpha_1 - 1 + \mathbf{E}[\omega_1(\varepsilon^r)]}}_{a_1^N} r^* + \underbrace{\frac{\sigma_r}{\alpha_1}}_{b_1} \varepsilon_t^r, \quad (11)$$

where I have defined the “endogenous switching” notation a_i^N for later use. Also, I have collected terms into

$$\omega_i(\varepsilon_t^r) = \frac{(\alpha_j - \alpha_i)p_{ij}(\varepsilon_t^r)}{\alpha_i(\alpha_j - 1) + \alpha_i \mathbf{E}[p_{ji}(\varepsilon^r)] + \alpha_j \mathbf{E}[p_{ij}(\varepsilon^r)]}, \quad i = 0, 1, j = 1 - i,$$

where the constants $\mathbf{E}[p_{ij}(\varepsilon^r)]$ are to be evaluated numerically in Appendix A.2, and $\mathbf{E}$ denotes unconditional expectation which is unrelated to time periods, hence time subscript of ε_t^r is omitted in $\mathbf{E}[p_{ij}(\varepsilon^r)]$ and $\mathbf{E}[\omega_i(\varepsilon^r)]$ for simplicity. Notice that $\omega_i(\varepsilon_t^r)$ depends on ε_t^r only through the $p_{ij}(\varepsilon_t^r)$ term in its numerator. So, $\mathbf{E}[\omega_i(\varepsilon^r)]$ can be readily obtained by replacing $p_{ij}(\varepsilon_t^r)$ in the numerator with its expectation $\mathbf{E}[p_{ij}(\varepsilon^r)]$. Comparing $\omega_i(\varepsilon_t^r)$ with $\mathbf{E}[\omega_i(\varepsilon^r)]$ for $i = 0, 1$, we can view $\omega_i(\varepsilon_t^r)$ term as the expectational impact of possible upcoming switch to regime j from regime i at t depending on current real rate shock through $p_{ij}(\varepsilon_t^r)$, and correspondingly $\mathbf{E}[\omega_i(\varepsilon^r)]$ term as the expectational impact of all future switching of regime beyond next period through $\mathbf{E}p_{ij}(\varepsilon^r)$. Intuitively, it justifies why the two terms appear separately in solution (10) and (11).

If we impose the restriction $\rho = 0$, regime switching becomes exogenous. In this case, expectations feedback mechanism is turned off and agents do not use real rate shock to forecast future regime change. Therefore, these naïve agents simply put constant weights on two regimes and the weights are measured by constant transition probabilities. In other words, the TVTPs collapse to two constants, i.e., $p_{ij}(\varepsilon_t^r)|_{\rho=0} =: p_{ij}$. Clearly, taking expectations would yield $\mathbf{E}p_{ij}(\varepsilon^r)|_{\rho=0} =$

⁸ Derivation is provided in Appendix A.

$p_{ij}(\varepsilon_t^r)|_{\rho=0} = p_{ij}$. Then, we have $\mathbf{E}[\omega_i(\varepsilon^r)] = \omega_i(\varepsilon_t^r)$ and, finally, the model solution rewrites as

$$\pi_{0t} = \underbrace{\frac{1 - \omega_0}{\alpha_0 - 1 + \omega_0}}_{a_0^X} r^* + \underbrace{\frac{\sigma_r}{\alpha_0}}_{b_0} \varepsilon_t^r,$$

and

$$\pi_{1t} = \underbrace{\frac{1 - \omega_1}{\alpha_1 - 1 + \omega_1}}_{a_1^X} r^* + \underbrace{\frac{\sigma_r}{\alpha_1}}_{b_1} \varepsilon_t^r,$$

where

$$\omega_i = \frac{(\alpha_j - \alpha_i)p_{ij}}{\alpha_i(\alpha_j - 1) + \alpha_i p_{ji} + \alpha_j p_{ij}}, \quad i = 0, 1, j = 1 - i, \quad (12)$$

and I have defined the “exogenous switching” coefficients a_i^X for latter use.

If we impose the restriction $\alpha_0 = \alpha_1 = \alpha$, solutions (10) and (11) collapse to the fixed-regime case, i.e.,

$$\pi_t = \underbrace{\frac{1}{\alpha - 1}}_{a^F} r^* + \underbrace{\frac{\sigma_r}{\alpha}}_b \varepsilon_t^r, \quad (13)$$

where I define the “fixed-regime” coefficients a^F and b . Accordingly, I define $a_0^F = 1/(\alpha_0 - 1)$ and $a_1^F = 1/(\alpha_1 - 1)$ with different α_0 and α_1 .

It is informative to compare the relations among a_i^F , a_i^X and a_i^N . First, if we assume that $\alpha_1 > \alpha_0 > 1$, we immediately get $a_1^F < a_0^F$. It simply implies that inflation is more stabilized if the central bank changes the policy rate more aggressively to inflation. Second, based on the same assumption $\alpha_1 > \alpha_0 > 1$, from (12) we can see that $\omega_0 > 0$ and $\omega_1 < 0$ since the denominators of ω_i are positive and the numerators depend on $\alpha_j - \alpha_i$. It then leads to $a_1^F < a_1^X$ and $a_0^X < a_0^F$. The intuition is clear. Compared to the case that policy regime is fixed at $\alpha = \alpha_1$, the expectations of policy switching to a less aggressive regime amplify the inflationary effect of real interest rate, hence the spillover between regimes leads to $a_1^F < a_1^X$. Vice versa for $a_0^X < a_0^F$.

Third, it is also straightforward to derive that $a_1^X < a_0^X$ provided the condition $(1 - \alpha_0)p_{11} + (1 - \alpha_1)p_{00} + \alpha_0\alpha_1 > 1$ holds. Davig and Leeper (2007) has derived this same result from their exogenous switching setup and named the condition as long-run Taylor principle. It is worthwhile to mention that as we depart from the fixed-regime case, we no longer need to assume both α_1 and α_0 to be larger than one, which is replaced by the long-run Taylor principle. The result $a_1^X < a_0^X$ indicates a larger inflation variability under the less aggressive regime. To compare a_1^N and a_0^N , I compute their difference and normalize by its exogenous counterpart $a_1^X - a_0^X$ and obtain

$$\frac{a_0^N - a_1^N}{a_0^X - a_1^X} = 1 - \frac{p_{01}(\varepsilon_t^r) - \mathbf{E} p_{01}(\varepsilon^r)}{\alpha_0} - \frac{p_{10}(\varepsilon_t^r) - \mathbf{E} p_{10}(\varepsilon^r)}{\alpha_1}. \quad (14)$$

The resulting measure is a random variable driven by the real rate shock. It measures the normalized dynamic spillover between two regimes in the endogenous switching case. The expected value of this spillover measure is one. It indicates that endogenous switching reduces to its exogenous counterpart and has the same level of spillover between regimes. If agents believe that the switching probability $p_{01}(\varepsilon_t^r)$ or $p_{10}(\varepsilon_t^r)$ is particularly large given the real rate shock ε_t^r , this measure can be sufficiently close to zero. That means, the inflationary effect of real interest rate can be as much amplified in the regime where policy is more aggressive as the less aggressive regime due to substantial expectational effects. On the contrary, if agents are confident that regime will not change, i.e., $p_{01}(\varepsilon_t^r)$ and $p_{10}(\varepsilon_t^r)$ are close to zero, then this measure becomes larger than one. The two regimes under the endogenous case are more apart compared to their exogenous counterpart. The spillover between regimes is much weaker in this situation. The variable spillover reflects the dynamic impact of expectations formation.

2.3 Dynamic Expectations Formation

I adapted the endogenous expectations formation effects from Leeper and Zha (2003). There are two differences between our definitions. First, they focus on the agents' expectations formation induced by policy intervention. My paper allows all policy and non-policy shocks to generate the expectational effect through future regime change. Second, as have been emphasized, the expectations formation effects are static in their setup, while the expectations feedback mechanism in my model produces dynamic, endogenous expectations formation effect.

To begin with, total effects and direct effects of a given shock are responses computed from the endogenous regime switching model and its fixed-regime counterpart. Both effects are reported relative to baseline inflation obtained from the fixed-regime model without conditioning on the given shock. Suppose the economy is under the less aggressive regime with α_0 at period t . Given a real rate shock ε_t^r , using solution (13) and (10), we write

$$\text{Direct Effects} = a_0^F r^* + b_0 \varepsilon_t^r - \text{Baseline},$$

$$\text{Total Effects} = a_0^N r^* + b_0 \varepsilon_t^r - \text{Baseline},$$

where $\text{Baseline} = a_0^F r^*$. Then, I minus direct effects from total effects and isolate endogenous expectations formation effects as

$$\text{Endogenous Expectations Formation Effects} = (a_0^N - a_0^F) r^*.$$

Analogously, using exogenous switching setup, I calculate exogenous expectations formation effects as

$$\text{Exogenous Expectations Formation Effects} = (a_0^X - a_0^F) r^*.$$

Figure 2. Effects of a Real Rate Shock on Inflation in Different Fisherian Models

Note: In Panel A, the dotted-dashed line indicates the contemporaneous inflation response under a fixed monetary policy rule with a less aggressive regime of $\alpha_0 = 1.5$. The solid line indicates that under an endogenous switching rule with $(\alpha_1, \rho, \phi, \delta) = (3, 0.9, 0.9, 0)$. The dashed line indicates an exogenous switching rule with $\rho = 0$. Additional parameters are set to be $(r^*, \sigma_r) = (2, 1)$. Panel B plots the expectations formation effects under endogenous and exogenous switching. Panel C reports their ratio. The different transition probabilities are displayed in Panel D.

These effects can be better demonstrated in Figure 2. Let $\alpha_0 = 1.5$. Panel A shows that a positive real rate shock would increase current inflation under a fixed-regime monetary policy rule. These changes are the direct effects of the shock. If monetary policy is allowed to switch to a more aggressive regime with $\alpha_1 = 3$ under exogenous switching, the increase of inflation is dampened by the exogenous expectations formation effect. However, it fails to capture the changing strength of the agents' belief according to the size of real rate shock. Under endogenous switching, when there is a larger real rate shock, agents form a stronger expectation of switching to the more aggressive regime. It translates into a much larger expectations formation effect that contributes to significant price stabilization. We can see that under a positive real rate shock of less than two standard

deviations, the endogenous expectations formation effect almost fully mitigates the shock’s direct impact on inflation. I depict the expectations formation effect, the equilibrium inflation under switching rules (dashed red and solid blue lines) minus that under the fixed-regime rule (the black o’s), in Panel B. Then, in Panel C, I calculate the ratio of expectational effects resulted from the endogenous-switching to that from the exogenous counterpart. For example, with a five standard deviation positive shock, the endogenous expectations formation effect is 2.91 times of its exogenous counterpart. In Panel D, I plot the transition probability to regime 1 under the two models. It is clear that the fluctuations of transition probability under endogenous switching contribute to the dynamic expectations formation. Overall, the strikingly large endogenous expectations formation effect depicted in Panel A resonates with Richard H. Clarida’s comment in Davig et al. (2006). Richard H. Clarida suggests that a less aggressive policy can nearly have the same effect as a more aggressive policy against an inflationary shock as long as it commits to switch regime when inflation gets out of control.

Additionally, Panel A displays that endogenous switching generates a favorable asymmetric and nonlinear feature. Notice that all asymmetry and nonlinearities stem from the endogenous switching policy process since the rest of the model economy is symmetric and linear. It becomes clear when we notice that the asymmetry and nonlinearity disappear in both exogenous switching and fixed-regime models. This result resonates with Davig et al. (2006) in that endogenous switching can cause symmetric fundamental shocks to generate asymmetric macroeconomic fluctuations, a feature documented in the literature.

In this section, by solving the threshold-switching Fisherian model analytically, I show that the expectations feedback mechanism creates a channel where real rate shocks alter the private sector’s confidence in future regime fluctuations. This dynamic expectation formation has profound consequences on economic dynamics. I demonstrate that the endogenous expectations formation effect is quantitatively much more significant than its exogenous counterpart and features asymmetric and nonlinear effects from real rate shocks. Thus, without taking the private sector’s evolving expectations into account, the evaluation of inflation is inaccurate, and policy advice from using exogenous switching models can be potentially misleading.

3 Threshold-Switching Monetary Policy in a New Keynesian Model

This section embeds a threshold-switching monetary policy process into a small-size New Keynesian model with benchmark specifications for the private sector. The New Keynesian model has become a conventional device to analyze the monetary policy as elaborated in Woodford (2003) and Galí (2015). My model follows the specifications in An and Schorfheide (2007) closely. The new element added is that the monetary policy process is subject to a threshold-type switching that links the current monetary policy regime to the fundamental historical shocks by an autoregressive regime

strength variable. The switching appears in responsiveness coefficients, smoothing motive, and the inflation target in the monetary policy behavior. Then, to solve the regime-switching DSGE model, I adopt a first-order perturbation method developed in Barthélemy and Marx (2017).

3.1 The Model

The economy describes the optimizing behavior of household and firms. It also features a monetary and fiscal authority. For simplicity, I abstract from capital and investment accumulation, which does not change qualitative results, as Clarida, Galí and Gertler (1999) claims.

The Representative Household The household chooses consumption good, work hours, and bond holding, $\{C_{t+s}, N_{t+s}, B_{t+s}\}_{s=0}^{\infty}$, to maximize expected lifetime utility

$$\mathbf{E}_t \sum_{s=0}^{\infty} d_{t+s} \beta^s \left(\frac{(C_{t+s}/A_{t+s})^{1-\epsilon} - 1}{1-\epsilon} - N_{t+s} \right),$$

where $\epsilon > 0$ and $\beta \in (0,1)$ are the relative risk aversion and the subjective discount rate that govern the household behavior, d_t is a preference shock influencing the household's intertemporal trade-offs, and A_t is a technology process.⁹ The intertemporal budget constraint is

$$P_t C_t + B_t + T_t \leq R_{t-1} B_{t-1} + P_t W_t N_t + P_t D_t.$$

Here, P_t denotes the price of consumption good, T_t is the lump-sum taxes, R_{t-1} is the gross interest rate for one-period government bond, W_t is the real hourly wage, and D_t is the real profit collected from owning firms.

The Representative Final Firm The firm is equipped with a “packaging technology”, or Dixit-Stiglitz aggregator, that can combine intermediate goods $Y_{\nu t}$ labelled as $\nu \in [0, 1]$, with a stochastic elasticity of substitution $\theta_t > 1$ between each other, into a final good, i.e.,

$$Y_t = \left[\int_0^1 Y_{\nu t}^{\frac{\theta_t-1}{\theta_t}} d\nu \right]^{\frac{\theta_t}{\theta_t-1}}.$$

In the assumed perfectly competitive final-good market, the firm takes price P_t as given and, to maximize profit, purchases an optimal amount of each intermediate good solved as

$$Y_{\nu t} = \left(\frac{P_{\nu t}}{P_t} \right)^{-\theta_t} Y_t, \tag{15}$$

⁹ The utility function exhibits habit formation. Current consumption enters the function relative to a habit stock measured by the nonstationary technology level. That is, only when consumption growth is catching up with technological progress, it generates positive utility.

where $P_{\nu t}$ is the price for the intermediate good. This expression is the demand function for each intermediate good. The zero-profit condition implies that the given final-good price P_t must be

$$P_t = \left[\int_0^1 P_{\nu t}^{1-\theta_t} d\nu \right]^{\frac{1}{1-\theta_t}}.$$

The Intermediate Firms Each intermediate-good-producing firm ν uses labor input $N_{\nu t}$ to produce output $Y_{\nu t}$ with a simple linear production function given by

$$Y_{\nu t} = A_t N_{\nu t}, \quad (16)$$

where A_t is homogenous across all firms. These firms can hire any amount of labor at the given hourly wage in a perfectly competitive labor market. In an assumed monopolistic competitive intermediate-goods market, the firm ν chooses the price and labor input, $\{P_{\nu,t+s}, N_{\nu,t+s}\}_{s=0}^{\infty}$, to maximize projected real profits

$$\mathbf{E}_t \sum_{s=0}^{\infty} d_{t+s} \beta^s \frac{MU_{t+s}}{MU_t} D_{\nu,t+s}$$

subject to its good $Y_{\nu t}$ demanded given in (15). Here, MU_t represents the household's marginal utility of the period- t consumption good and equals to $C_t^{-\epsilon}/A_t^{1-\epsilon}$. Hence, the combined term $d_{t+s} \beta^s \frac{MU_{t+s}}{MU_t}$ measures the firm's exogenous stochastic discount factor. Meanwhile, real profit $D_{\nu t}$ can be calculated as

$$D_{\nu t} = \frac{P_{\nu t} Y_{\nu t}}{P_t} - MC_t Y_{\nu t} - \frac{\psi}{2} \left(\frac{P_{\nu t}}{\Pi_{s_t}^* P_{\nu,t-1}} - 1 \right)^2 Y_{\nu t},$$

where $MC_t = W_t/A_t$ is the real marginal cost of production derived from (16). Following Rotemberg (1982), the last term captures the real adjustment cost when firm ν adjusts its price. Here, $\psi \geq 0$ denotes the level of price stickiness common to all firms, $\Pi_{s_t}^*$ is the steady-state gross rate of inflation measured by final-good price at regime s_t that concurs with the Fed's regime-dependent inflation target. Firm ν 's optimal price setting implies that its desired markup over its marginal cost can be determined as

$$u_t = \frac{\theta_t}{\theta_t - 1}.$$

Correspondingly, the steady-state markup is $u = \frac{\theta}{\theta-1}$, where θ denotes the steady-state elasticity of substitution.

Fiscal Policy and Market Clearing The government spending G_t is given as a stochastic fraction ζ_t of the output Y_t . That is, $G_t = \zeta_t Y_t$. To achieve its budgetary solvency, the government

passively issues the nominal government bonds and adjusts lump-sum taxes.¹⁰ Therefore, the government budget constraint writes as

$$P_t G_t + R_{t-1} B_{t-1} = T_t + B_t.$$

The goods market clears when production equals consumption, government expenditure plus a real adjustment cost as

$$Y_t = C_t + G_t + \frac{\psi}{2} \left[\frac{\Pi_t}{\Pi_{s_t}^*} - 1 \right]^2 Y_t. \quad (17)$$

Monetary Policy The monetary authority adjusts the policy rate following

$$\frac{R_t}{R_{s_t}^*} = \left[\left(\frac{\Pi_t}{\Pi_{s_t}^*} \right)^{\alpha(s_t)} \left(\frac{Y_t}{Y_t^*} \right)^{\gamma(s_t)} \right]^{1-\phi_R(s_t)} \left(\frac{R_{t-1}}{R_{s_{t-1}}^*} \right)^{\phi_R(s_t)} e^{\sigma_e \varepsilon_t^e}, \quad (18)$$

where Π_t denotes the gross inflation given as P_t/P_{t-1} , $\alpha(s_t)$ and $\gamma(s_t)$ capture the switching responsiveness of the policy rate to the inflation and output gaps, $\phi_R(s_t)$ measures the switching smoothing motive of the policymaker who sets the nominal rate. The regime indicator s_t governs the switching of these parameters. $R_{s_t}^*$ is the regime-dependent steady-state nominal interest rate and Y_t^* is the prevailing output level absent nominal rigidities and markup shock, both will be computed in Appendix B. Finally, ε_t^e denotes the monetary policy shock and $\varepsilon_t^e \sim \mathcal{N}(0, 1)$. Taking logarithm of (18) yields

$$\hat{R}_t = (1 - \phi_R(s_t)) [\alpha(s_t) \hat{\Pi}_t + \gamma(s_t) \hat{Y}_t] + \phi_R(s_t) \hat{R}_{t-1} + \sigma_e \varepsilon_t^e.$$

The variables with a hat represent the log-deviation from the steady-state level or from a trend path in the case of output. The regime indicator s_t is determined by an autoregressive regime strength index s_t^* as

$$s_t = \mathbb{1}\{s_t^* \geq 0\}, \quad (19)$$

and the regime index s_t^* evolves as an AR(1) process specified as

$$s_{t+1}^* = (1 - \phi) \delta + \phi s_t^* + w_{t+1}. \quad (20)$$

Shocks Fluctuations of the economy are driven by five fundamental shocks and one non-fundamental shock. Aggregate productivity A_t grows at a constant rate and subjects to an exogenous fluctuation

¹⁰ See, for example, Bianchi and Ilut (2017) for analysis of a regime-switching DSGE model with switches in both monetary and fiscal policies. They also allow for two volatility regimes for structural shocks following Sims and Zha (2006).

a_t , specified as

$$A_t = e^\lambda a_t A_{t-1}, \quad (21)$$

where λ is the average technology net growth rate. Let $g_t = 1/(1 - \zeta_t)$. Then, the shocks to technology, preference, markup, and government spending are given by

$$\ln a_t = \phi_a \ln a_{t-1} + \sigma_a \varepsilon_t^a, \quad (22)$$

$$\ln d_t = \phi_d \ln d_{t-1} + \sigma_d \varepsilon_t^d, \quad (23)$$

$$\ln u_t - \ln u = \phi_u (\ln u_{t-1} - \ln u) + \sigma_u \varepsilon_t^u, \quad (24)$$

$$\ln g_t - \ln g = \phi_g (\ln g_{t-1} - \ln g) + \sigma_g \varepsilon_t^g, \quad (25)$$

where $|\phi_k| < 1$, for $k \in \{a, d, u, g\}$, and g is the steady-state government spending. The five disturbances $\varepsilon_t = (\varepsilon_t^a, \varepsilon_t^d, \varepsilon_t^u, \varepsilon_t^e, \varepsilon_t^g)'$ are independent of each other and each is i.i.d. standard normal.

Additionally, the endogenous regime strength index brings in a regime index innovation w_{t+1} , which is distributed as multivariate normal together with the past fundamental shocks ε_t , viz.,

$$\begin{pmatrix} \varepsilon_t^a \\ \varepsilon_t^d \\ \varepsilon_t^u \\ \varepsilon_t^e \\ \varepsilon_t^g \\ w_{t+1} \end{pmatrix} \sim \mathcal{N} \left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \rho_{aw} \\ 0 & 1 & 0 & 0 & 0 & \rho_{dw} \\ 0 & 0 & 1 & 0 & 0 & \rho_{uw} \\ 0 & 0 & 0 & 1 & 0 & \rho_{ew} \\ 0 & 0 & 0 & 0 & 1 & \rho_{gw} \\ \rho_{aw} & \rho_{dw} & \rho_{uw} & \rho_{ew} & \rho_{gw} & 1 \end{pmatrix} \right), \quad (26)$$

where $\rho = (\rho_{aw}, \rho_{dw}, \rho_{uw}, \rho_{ew}, \rho_{gw})'$, with the normalization $\rho' \rho < 1$. Equilibrium relations of the model is summarized in Appendix B.

Projecting w_{t+1} onto ε_t yields

$$w_{t+1} = \rho' \varepsilon_t + \sqrt{1 - \rho' \rho} v_{t+1}, \quad v_{t+1} \sim \mathcal{N}(0, 1),$$

where ε_t and v_{t+1} are orthogonal and each has unit variance. Hence, equation (20) is further specified as

$$s_{t+1}^* = (1 - \phi) \delta + \phi s_t^* + \rho' \varepsilon_t + \sqrt{1 - \rho' \rho} v_{t+1}.$$

Now, note that

$$s_{t+h}^* = (1 - \phi^h) \delta + \phi^h s_t^* + \sum_{j=1}^h \phi^{h-j} (\rho' \varepsilon_{t+j-1} + \sqrt{1 - \rho' \rho} v_{t+j}),$$

for $h \geq 1$. Therefore, the decomposition of h -step forecast error variance (FEV) for s_t^* can be expressed as

$$\begin{aligned}
FEV[s_t^*(h)] &= \mathbf{Var} [s_{t+h}^* - \mathbf{E}(s_{t+h}^* | \mathcal{F}_t)] \\
&= \sum_{k=1}^5 \underbrace{\sum_{n=2}^h \phi^{2(h-n)} \rho_k^2}_{\text{by } k\text{-th fundamental}} + \underbrace{\sum_{n=1}^h \phi^{2(h-n)} (1 - \rho' \rho)}_{\text{by non-fundamental}} \\
&= \frac{1}{1 - \phi^2} + \left(1 - \rho' \rho - \frac{1}{1 - \phi^2}\right) \phi^{2(h-1)} \\
&\rightarrow \frac{1}{1 - \phi^2}, \quad \text{for } h \rightarrow \infty,
\end{aligned} \tag{27}$$

where $\mathcal{F}_t$ includes entire history for all variables except for (s_t^*) and (v_t) . Then, it is straightforward to compute that, in the limit, the contribution of the k -th fundamental shock to the forecast error variance of monetary policy regime strength index is ρ_k^2 . Consequently, $1 - \rho' \rho$ captures the contribution of the non-fundamental regime index shock.

3.2 Numerical Solutions

I solve the resulting endogenous regime-switching monetary DSGE model using a first-order perturbation method, which largely facilitates the estimation. Maih (2015) and Foerster, Rubio-Ramrez, Waggoner and Zha (2016) develop perturbation solutions to regime-switching DSGE models following the successive differentiation idea of Kim, Kim, Schaumburg and Sims (2008). More recently, Barthélemy and Marx (2017) create a new perturbation algorithm using a functional iteration technique based on Cho (2016). Moreover, they apply the implicit function theorem to numerically ensure determinacy condition and accuracy of the approximate solution. My paper uses Barthélemy and Marx (2017)'s approach.

A prerequisite for applying Barthélemy and Marx (2017)'s perturbation algorithm is to cast the threshold-type switching into a time-varying transition probabilities representation. Following the steps summarized in Appendix A.2, the transition probability from regime 0 to regime 0 can be computed as

$$p_{00}(\varepsilon_t) = \frac{\int_{-\infty}^{-\delta\sqrt{1-\phi^2}} \Phi\left(\frac{-\delta - \rho'\varepsilon_t}{\sqrt{1-\rho^2}} - \frac{\phi z}{\sqrt{(1-\phi^2)(1-\rho^2)}}\right) \varphi(z) dz}{\Phi(-\delta\sqrt{1-\phi^2})}, \tag{28}$$

where $\Phi(\cdot)$ and $\varphi(\cdot)$ denote the standard normal distribution's cumulative distribution function

(CDF) and probability density function (PDF). Similarly, we can compute

$$p_{10}(\varepsilon_t) = \frac{\int_{-\delta\sqrt{1-\phi^2}}^{\infty} \Phi\left(\frac{-\delta - \rho'\varepsilon_t}{\sqrt{1-\rho^2}} - \frac{\phi z}{\sqrt{(1-\phi^2)(1-\rho^2)}}\right) \varphi(z) dz}{\Phi(\delta\sqrt{1-\phi^2})}. \quad (29)$$

Then, the other two transition probabilities are simply given as $p_{01}(\varepsilon_t) = 1 - p_{00}(\varepsilon_t)$ and $p_{11}(\varepsilon_t) = 1 - p_{10}(\varepsilon_t)$. With these TVTPs, I rewrite the full characterization in Appendix B as a set of equations consisting of equilibrium conditions (50)-(51), (17)-(18), exogenous shocks (21)-(25), and the TVTPs (28)-(29).

To obtain stationarity, it is convenient to detrend output and consumption against the technology process A_t . So, we have $c_t = C_t/A_t$, $y_t = Y_t/A_t$. I define a regime-dependent steady state as an equilibrium when shocks are turned off and inflation is at its regime-dependent target rate. Then, the regime-dependent steady states regarding the detrended variables are

$$X_{s_t}^* = (y^*, \Pi_{s_t}^*, R_{s_t}^*, a^*, d^*, u^*, g^*)' = \left(g \left(\frac{\theta - 1}{\theta} \right)^{1/\epsilon}, \Pi_{s_t}^*, \frac{e^\lambda}{\beta} \left(\frac{\bar{p}_{s_t,0}}{\Pi_0^*} + \frac{\bar{p}_{s_t,1}}{\Pi_1^*} \right)^{-1}, 1, 1, u, g \right)',$$

where $\bar{p}_{s_t,0}$ and $\bar{p}_{s_t,1}$ are transition probabilities from s_t to regime 0 and 1 when shocks are turned off.¹¹ c_t disappears as a result of substitution with y_t using the market clearing condition (17). I rewrite all variables as log-deviations from their regime-dependent steady states which are signified by ‘hat’ and collected in a (7×1) vector x_t as $(\hat{y}_t, \hat{\Pi}_t, \hat{R}_t, \hat{a}_t, \hat{d}_t, \hat{u}_t, \hat{g}_t)'$. So, I can express my model as

$$\mathbf{E}_t[f_{s_t}(x_{t+1}, x_t, x_{t-1}, \chi\varepsilon_t)] = 0, \quad (30)$$

where χ is an inserted scale parameter. Let the set U^t represents an entire history of states and shocks, i.e., $(s_{0:t}, \varepsilon_{0:t}) = (s_n, \varepsilon_n)_{n=0}^t$. Then, a model solution is a bounded, continuous mapping $m : U^t \rightarrow \mathbb{R}^7$ from these regime states and shocks to vector x_t , which satisfies the condition (30) specified by the full characterization provided in Appendix C. Let $\mathcal{M}$ denotes the set of bounded and continuous functions m on U^t . The left-hand side of (30) represents an operator $\mathbf{T}$ that acts upon such a function m and the scale parameter χ . I may therefore rewrite (30) as

$$\mathbf{T}(m, \chi)(s_{0:t}, \varepsilon_{0:t}) = \sum_{j=0}^1 \int p_{s_t,j}(\varepsilon_t) f_{s_t}(m(j, \varepsilon, s_{0:t}, \varepsilon_{0:t}), m(s_{0:t}, \varepsilon_{0:t}), m(s_{0:t-1}, \varepsilon_{0:t-1}), \chi\varepsilon_t) \varphi(\varepsilon) d\varepsilon = 0,$$

where $\varphi(\varepsilon)$ denotes the normal probability density function for ε . In this functional framework, solving the original model is rephrased as the root-finding problem of the operator equality $\mathbf{T}(m, \chi) = 0$.

Barthélemy and Marx (2017) applies the implicit function theorem to compute this solution.

¹¹ Appendix B provides derivations for steady states.

The steady state of the log-deviation variables in model (30) is simply given by $x^* = 0$. By the implicit function theorem, the first-order perturbation solution around the steady-state $x^* = 0$, with χ set at one, is given by

$$x_t = \underbrace{A(s_t, \Theta)}_{7 \times 7} x_{t-1} + \underbrace{B(s_t, \Theta)}_{7 \times 5} \varepsilon_t, \quad (31)$$

where each of the four coefficient matrices $A(s_t = i)$ and $B(s_t = i)$ for $i = 0, 1$ depends on all the DSGE parameters, and the parameters (ϕ, δ, ρ') defining the TVTPs, collected in Θ . The analytical solution (9) suggests that the TVTP would enter true solution as a nonlinear function of the five fundamental shocks through matrices $A(s_t, p_{s_t,0}(\varepsilon_t))$. But here the first-order linear approximation of the TVTP, therefore, the endogenous expectations formation effect, with respect to the five fundamental shocks, enter the numerical solution as a portion of $B(s_t)\varepsilon_t$. As demonstrated in Figure 2, a small shock can induce substantial endogenous expectations formation effects. It justifies that the first-order perturbation solution preserves the endogeneity of regime-switching.

4 Estimation of the Threshold-Switching New Keynesian Model

4.1 Estimation Strategy

4.1.1 Data

To construct the system of measurement equations specified below in (32), I need three series of U.S. quarterly data. All data are downloaded from FRED, Federal Reserve Bank of St. Louis (FRED mnemonics in parentheses). For the growth rate of real output per capita (in percent) YGR_t , I divide the quarterly real gross domestic product ($GDPC1$) by the quarterly average of the civilian non-institutional population ($CNP16OV$) to obtain real output per capita, then take its log difference and scale by 100. For the annualized net inflation rate (in percent) INF_t , I take a log difference of quarterly consumer price index ($CPIAUCSL$) and scale by 400. For the annualized net nominal interest rate (in percent) INT_t , I use the effective federal funds rate ($FEDFUNDS$). The sample spans 1954:Q3 to 2007:Q4. The sample means and sample standard deviations (both in percent) of the quarterly series (YGR_t, INF_t, INT_t) are (0.47, 3.84, 5.69) and (0.90, 2.96, 3.30), respectively. I do not include post-crisis data because the prototypical New Keynesian model can not well explain the monetary policy carried out near the zero lower bound.¹²

¹² Recent studies about monetary policy at the zero lower bound using nonlinear DSGE models can be found in Bianchi and Melosi (2017), Gust, Herbst, López-Salido and Smith (2017), and Boragan Aruoba, Cuba-Borda and Schorfheide (2018), among others.

4.1.2 State Space Representation

In Appendix C, I develop an efficient threshold-switching Kalman filter to estimate the threshold-type endogenous-switching state space model. For estimation, I first augment the state vector with lagged $\hat{y}_{t-1}$ and the shock vector ε'_t , and define the augmented state vector as $\varsigma_t = (x'_t, \hat{y}_{t-1}, \varepsilon'_t)'$. Therefore, the measurement equation is given as

$$f_t = D(s_t, \Theta) + Z(\Theta)\varsigma_t + \Sigma_\nu^{1/2}\nu_t,$$

which can be more explicitly written as

$$\underbrace{\begin{pmatrix} YGR_t \\ INF_t \\ INT_t \end{pmatrix}}_{f_t} = \underbrace{\begin{pmatrix} \lambda^{(Q)} \\ \pi_{s_t}^{(A)} \\ 4\lambda^{(Q)} + r^{(A)} + \pi_{s_t}^{(A)} \end{pmatrix}}_{D(s_t, \Theta)} + 100 \underbrace{\begin{pmatrix} \hat{y}_t - \hat{y}_{t-1} + \hat{a}_t \\ 4\hat{\Pi}_t \\ 4\hat{R}_t \end{pmatrix}}_{Z(\Theta)\varsigma_t} + \Sigma_\nu^{1/2}\nu_t, \quad (32)$$

where $\lambda^{(Q)}$ is the quarterly net technology growth rate in percent, $r^{(A)}$ the annualized net real interest rate in percent, and $\pi_{s_t}^{(A)}$ the annualized net inflation target in percent. They are connected with model parameters as $(\lambda^{(Q)}, r^{(A)}, \pi_{s_t}^{(A)}) = (100(e^\lambda - 1), 400(1/\beta - 1), 400(\Pi_{s_t}^* - 1))$. $\Sigma_\nu^{1/2}\nu_t$ is the measurement error following multivariate normal distribution with covariance matrix as Σ_ν . Σ_ν is preassigned so that the standard deviation of each measurement error equals 20% standard deviation of its corresponding observable. Transition equations consist of solution (31), and two identities $\hat{y}_{t-1} = \hat{y}_{t-1}$ and $\varepsilon_t = \varepsilon_t$, which is given by

$$\varsigma_t = \underbrace{M(s_t, \Theta)}_{13 \times 13} \varsigma_{t-1} + \underbrace{G(s_t, \Theta)}_{13 \times 5} \varepsilon_t, \quad (33)$$

where

$$M(s_t, \Theta) = \begin{pmatrix} \underbrace{A(s_t)}_{7 \times 7} & \underbrace{0}_{7 \times 6} \\ 1 & \underbrace{0}_{1 \times 12} \\ & \underbrace{0}_{5 \times 13} \end{pmatrix}, \quad G(s_t, \Theta) = \begin{pmatrix} \underbrace{B(s_t)}_{7 \times 5} \\ \underbrace{0}_{1 \times 5} \\ \underbrace{I_5}_{5 \times 5} \end{pmatrix}.$$

For the above state space representation (32) and (33), the likelihood function can be efficiently evaluated using my newly-developed filter. Then, I adopt a Bayesian approach for the estimation. Specifically, priors are augmented to induce some desired curvature in the likelihood function surface. Posterior samples are drawn from the Markov Chain Monte Carlo (MCMC) sampler with a standard random walk Metropolis-Hastings algorithm.

4.2 Empirical Analysis

4.2.1 Prior Distribution

Priors are reported in the third to fifth columns of Table 1. All priors, except for those parameters characterizing the expectations feedback mechanism, are in line with existing literature. First, coefficient to output gap γ and smoothing motive of federal funds rate ϕ_R are from Bianchi (2012), risk aversion ϵ and quarterly technology growth rate $\lambda^{(Q)}$ are the same as An and Schorfheide (2007), and the slope of New Keynesian Phillips curve (NKPC) κ is adopted from Davig and Doh (2014).¹³ Second, the coefficients to inflation α_0, α_1 is close to the priors in Davig and Doh (2014). I specify the difference between the two coefficients $\alpha_1 - \alpha_0$ as a gamma distribution so that it guarantees $\alpha_1 > \alpha_0$ and hence identifying the regimes. Annualized net inflation target $\pi^{(A)}$ is similar to that in An and Schorfheide (2007). Annualized net real interest rate $r^{(A)}$ is adopted from Bianchi (2012) with a modified prior mean. This prior mean implies a value of β equaling 0.999 that is close to the prior mean of β used in Davig and Doh (2014). Shock standard deviations and autocorrelation coefficients are relatively loose, given as Inverse Gamma(2,0.1) and Beta(0.5,0.2). Third, the newly-introduced parameters (ρ', ϕ, δ) have relatively loose priors. For instance, ρ' are all set to be uniformly distributed between -1 and 1. Finally, notice that, the steady-state government spending g and intermediate goods' elasticity of substitution θ do not affect the first-order perturbation solution, hence not estimated.

4.2.2 Posterior Estimates

I draw 500,000 MCMC samples from the posterior distribution and burn-in the first 100,000 samples. Based on the remaining 400,000 draws, I compute the posterior mean and the 5th and 95th percentiles for all parameters and report them in Table 1. The acceptance rate of the MCMC draws is 17.6%, which is relatively close to the suggested optimal rate of 23.4%. It indicates that the MCMC draws represent the high-dimensional posterior distribution nicely. On the other hand, Figure 9, 10, and 11 of the recursive means, trace plots, and autocorrelation functions of the MCMC draws for all parameters suggest that these Markov chains converge relatively well.

Regarding the monetary policy rule, inflation responsiveness are estimated to be 0.93 and 2.37 for less aggressive and more aggressive regimes, respectively. So, monetary policy is estimated to be passive in regime-0 since it responded less than one-to-one to inflation. In the less aggressive regime, the central bank's response to the output gap is stronger at 0.93, the smoothing motive of the federal funds rate is weaker at 0.52, and the annualized inflation target is lower at 3.34%. Vice versa for the more aggressive regime. Other parameter estimates are also consistent with existing

¹³ The slope of NKPC $\kappa = \frac{\epsilon(\theta-1)}{\psi\Pi_{s_t}^2}$ is regime-dependent. But $\Pi_{s_t}^{*2}$ at both regimes are very close to 1. So, I neglected the regime dependence of κ in my estimation.

work. For example, the annualized real interest rate is 0.47%, the quarterly technology growth rate is 0.39%, relative risk aversion is 2.94, and the slope of the NKPC is 0.08. I analyze the parameter

Table 1. Posterior Estimates and Priors of the Regime-Switching DSGE Model Parameters

Description	Parameter	Density	Prior		Posterior		
			Para(1)	Para(2)	Mean	5%	95%
response to inflation	α_0	G	1.00	0.20	0.93	0.68	1.19
diff in inflation responses	$\alpha_1 - \alpha_0$	G	1.00	0.30	1.44	1.06	1.88
response to output gap	γ_0	G	0.25	0.15	0.93	0.64	1.29
response to output gap	γ_1	G	0.25	0.15	0.67	0.48	0.90
FFR smoothing motive	ϕ_R^0	B	0.60	0.20	0.52	0.36	0.66
FFR smoothing motive	ϕ_R^1	B	0.60	0.20	0.82	0.77	0.86
inflation target	$\pi_0^{(A)}$	G	4.00	1.00	3.34	2.73	3.94
inflation target	$\pi_1^{(A)}$	G	4.00	1.00	4.04	3.49	4.56
real interest rate	$r^{(A)}$	G	0.50	0.20	0.47	0.22	0.79
technology growth rate	$\lambda^{(Q)}$	N	0.40	0.20	0.39	0.28	0.50
relative risk aversion	ϵ	G	2.00	0.50	2.94	2.22	3.79
slope of the NKPC	κ	G	0.50	0.20	0.08	0.05	0.12
<i>Exogenous Process Parameters:</i>							
technology	ϕ_a	B	0.50	0.20	0.63	0.51	0.73
preference	ϕ_d	B	0.50	0.20	0.87	0.81	0.91
markup	ϕ_u	B	0.50	0.20	0.92	0.88	0.95
government spending	ϕ_g	B	0.50	0.20	0.50	0.17	0.83
technology	$100\sigma_a$	IG	2.00	0.10	0.46	0.37	0.56
preference	$100\sigma_d$	IG	2.00	0.10	2.79	2.22	3.48
markup	$100\sigma_u$	IG	2.00	0.10	1.96	1.33	2.78
monetary policy	$100\sigma_e$	IG	2.00	0.10	0.06	0.04	0.08
government spending	$100\sigma_g$	IG	2.00	0.10	0.10	0.05	0.18
<i>Threshold Switching Parameters:</i>							
regime persistence	ϕ	B	0.90	0.05	0.90	0.85	0.94
average regime strength	δ	N	0.00	0.50	0.81	0.25	1.34
technology	ρ_{aw}	U	-1.00	1.00	0.36	0.03	0.72
preference	ρ_{dw}	U	-1.00	1.00	0.04	-0.29	0.34
markup	ρ_{uw}	U	-1.00	1.00	-0.81	-0.97	-0.55
monetary policy	ρ_{ew}	U	-1.00	1.00	0.03	0.00	0.10
government spending	ρ_{gw}	U	-1.00	1.00	0.12	-0.41	0.61

Note: I draw 500,000 samples from the posterior and burn-in the first 100,000 samples. Based on the remaining 400,000 samples, I compute the posterior mean and the 5th and 95th percentiles. The acceptance rate of the MCMC draws is 17.6%. Para(1) and Para(2) represent the left side and right side for the uniform (U) distribution, the parameters (ν, ι) in the inverse-gamma (IG) distribution specified as $p(x | \nu, \iota) \propto x^{-\nu-1} \exp(-\nu \iota^2 / 2x^2)$, and the mean and standard deviation for the gamma (G), beta (B), and normal (N) distributions.

estimates of those defining the expectations feedback mechanism in more detail. The estimate of δ , which is the unconditional mean of regime strength index, is positive at 0.81. It reflects that the more aggressive regime was more prevalent during the sample periods investigated. Regime strength index's autocorrelation coefficient ϕ is 0.90, which means the impact of shocks on the regime index is relatively persistent. More importantly, the endogeneity parameter ρ_{uw} is estimated to be -0.81, which is of the most massive magnitude among the five fundamental shocks driving monetary policy regime change. As I pointed out earlier below (27), ρ_k^2 captures the k -th shock's contribution to the forecasting error variance of monetary policy regime strength index. Therefore, 65.6% of U.S. monetary policy regime evolution is due to past markup shocks. The contribution from technology shock, government spending shock, preference shock, and monetary policy shock are 13.0%, 1.4%, 0.2%, and 0.1%, respectively. In particular, we are most confident that markup shocks have, while monetary policy shocks do not have a significant impact on the monetary policy regime change due to their narrower 90% credible intervals among the five shocks. On the other hand, the implied contribution from the non-fundamental regime index shock is 19.7%.

4.2.3 Filtered Monetary Policy Regime Strength Index

Panel 1 in Figure 3 shows the extracted policy regime strength index and the filtered regime-1 probabilities. Shaded bars denote recessions defined by the NBER. The extracted regime strength index and regime-1 probability are consistent in identifying monetary policy regimes. But, the regime index provides additional marginal information regarding the strength of prevailing regimes. This difference highlights the advantage of the regime index, which features strong persistence against the regime-1 probability. Persistence facilitates the forecasting of regime fluctuations.

Monetary policy is more aggressive from 1954:Q3 to 1970:Q4, except for a brief period in 1958:Q3-Q4 following the short recession of 1957:Q3-1958:Q2. During about the two decades, Fed Chairman Martin pursued policy objectives of low inflation and economic stability. In the famous words of Martin, the Fed's duty is "to take away the punch bowl just as the party gets going." That means the Fed would raise interest rates once the economy reaches full employment, and prices begin to increase too rapidly.

From the early 1970s, the policy regime index started to fall below zero, as can be seen clearly in the upper panel. For the first time, monetary policy entered a persistent and substantial less aggressive regime during 1973:Q4-1976:Q1. It climbed upwards above zero very briefly in 1976:Q2-Q3 and 1977:Q4-1978:Q1. Then, the second substantial and persistent less aggressive regime prevails until 1981:Q1 and saw a historic low level of regime index in 1980:Q3. The bottom panel shows clearly that, during this time, the movement of inflation is negatively correlated to the policy regime index. Observers often think that the Federal Reserve Chairman Arthur F. Burns is responsible for the Great Inflation of the 1970s. He has a reputation of being overly influenced by the political

Figure 3. Filtered Monetary Policy Regime Strength, Regime-1 Probability, and Inflation

Note: The shaded bars denote recessions defined by the NBER. Sample period spans from 1954:Q3 ~ 2007:Q4. In the upper panel, the blue solid line (correspond with left y-axis) displays the extracted latent regime strength index and the black horizontal line is at level zero. The red dashed line (correspond with right y-axis) reports the filtered probability of being in the more aggressive regime. The annualized quarterly total inflation rates (FRED mnemonic “*CPIAUCSL*”) and annualized quarterly core inflation rates (FRED mnemonic “*CPILFESL*”) are shown in the bottom panel. The core inflation data is only available since 1957:Q2.

pressure and being unwilling to quell inflation at the cost of raising unemployment rate. The filtered regime index shows that monetary policy was extremely easing for almost the entire duration of his chairmanship from 1970:Q1-1978:Q1. To achieve a tempered inflation would have required a much tigher monetary policy, which must have been politically very costly.

The regime strength index started to pick up from the historic low level in 1980:Q3, about a year after Paul Volcker was appointed as Fed Chairman in August 1979. The sustained less aggressive regime ended in 1983:Q1. When Volcker took office, high inflation has become a crucial economic and political challenge to the United States. The sharp rise in the regime index, which placed it significantly above zero, confirms the widely accepted belief that he successfully curbed inflation. Since 1983:Q1, the U.S. experienced a prolonged more aggressive regime until the end of 2007, except for a few temporary policy switches to the less aggressive regime in 1991:Q1, 2003:Q2, 2006:Q4, and 2007:Q3. We also see the regime index lingering at the threshold level at the end of the sample period, which marks the onset of the 2008 Great Recession.

Over the entire sample period, it becomes apparent that the monetary regime index correlates

negatively with inflation. The correlation coefficient is -0.66 . A possible implication of this negative correlation, together with the above narratives, is that: By and large, the dynamics of inflation are more likely to be the outcome, rather than the cause, of postwar U.S. monetary policy regime switches. In other words, if high inflations always trigger monetary policy to move into the more aggressive regime, as assumed in a simplified model studied in Davig et al. (2006), we should at least see some positive correlation between inflation and regime index. However, it is not in the post-WWII U.S. data. Indeed, the central bank has already taken inflation into account in their systematic response captured by the fixed-regime Taylor rule. As assumed in this paper, the switches of the regime are more likely policy response to non-target information beyond inflation and output gap.

Figure 4 reports the historical decomposition of filtered regime strength index against the five fundamental shocks and one regime index shock. First, these historical decompositions demonstrate that markup shock was the main driver of the U.S. monetary policy regime shifts during the post-World War II history, which is in line with the forecast error variance decomposition indicated by

Figure 4. Historical Decomposition of Extracted Regime Strength Index

Note: This figure plots the historical decomposition of extracted regime strength index against the five fundamental shocks and one regime index shock. In each subplot, the black line indicates the actual regime strength index, while the colored line indicates the regime strength less corresponding shock. For example, in the third subplot, if markup shock were turned off during the entire sample history, the actual regime strength (black line) would become the red line. The shaded bars denote recessions defined by the NBER. Data covers a time period from 1954:Q3 ~ 2007:Q4.

Figure 5. Filtered Shocks

Note: This figure plots the filtered values (blue lines) of five fundamental shocks and one regime index shock of the DSGE model. The red lines are the eight-period (two-year) simple moving average of the corresponding shocks. The shaded bars denote recessions defined by the NBER. Sample period spans from 1954:Q3 ~ 2007:Q4.

the estimates of vector ρ . It is followed by regime index shock, technology shock, and then the least significant ones, namely, government spending shock, preference shock, and monetary policy shock. Second, if there were no markup shock, the regime index would be much smaller in magnitude over the entire history, which means that the regime strength would be weaker. Moreover, eight out of the eleven less aggressive regime periods would not have happened without markup shock. Among the remaining three periods, the two persistent and substantial periods throughout the 1970s would be much weaker, becoming comparable to another period in 1958 regarding regime strength.

Filtered shocks and their two-year simple moving averages are shown in Figure 5. Two remarks arise. First, the sizable and long-lasting positive markup shocks during the 1970s translate into the much lower level of actual regime index (black line) compared to the higher regime index without markup shock (red line), as shown in the corresponding subplot in Figure 4. Second, regarding the sharp rise of regime index in the early 1980s, my estimation suggests that it is driven by the vanishing of adverse markup shock and a combination of all other five shocks with magnitudes of these shocks almost reaching their historic high.

4.2.4 The Fed Confronting Adverse Supply Shocks with Dual Mandate: Regime Switches and Inflation Expectations Management

The historical decomposition of regime index indicates that it is mainly adverse markup shocks that pull monetary policy into the less aggressive regime. We know that adverse markup shocks, like steep rise in oil prices in the 1970s, would lead to rising inflation and declining output, the so-called stagflation. It puts the central bank in a painful dilemma of achieving its dual mandate of stabilizing both inflation and output. The postwar U.S. data suggest that the Federal Reserve often responded to adverse markup shock by switching into the less aggressive regime. Relating to the optimal monetary policy literature, it seems to suggest that the monetary authority put a larger weight on the welfare gain of stabilizing output than stabilizing inflation when the economy faces adverse markup shocks.¹⁴

But what about inflation? Incumbent president and CEO of the Boston Fed offered his insights about this question in Rosengren et al. (2011) when he gave a speech to the private sector. I organize some of his views here. Adverse supply shocks often relate to initial increases in food and energy prices. It causes total inflation, which includes food and energy sectors, to rise. However, whether the impact would transmit to other sectors, measured by the core inflation, depends. The bottom panel of Figure 3 compares the fluctuation of total and core inflations. Rosengren then argues, whether the impact would pass onto prices in other sectors mostly depend on how agents' inflation expectations, $\mathbf{E}_t(\pi_{t+1})$, respond to the shock. If private sector expects that the Fed will not tolerate any long-term effect on inflation and food and energy prices will stabilize, i.e., the expectations to switch into or stay at the more aggressive regime, $p_{s_t,1}(\varepsilon_t^u)$, is high, then the rise in total inflation will likely decline and eventually gravitates to the lower core inflation. Bottom panel of Figure 3 shows that this is likely the case we saw in post-1986 periods as total inflations (blue line) were always attracted to core inflations (red line). Alternatively, if private agents believe the central bank will switch to or stay at the less aggressive regime, that is, $p_{s_t,0}(\varepsilon_t^u)$ is high, then inflation expectations, $\mathbf{E}_t(\pi_{t+1})$, do rise to the adverse supply shock. The expectations formation effect would then pressure the current prices of all other goods to increase. Eventually, the adverse supply shocks propagate through the economy and affect all prices. In this case, we would observe the core inflation converges to higher total inflation. As shown in the bottom panel of Figure 3, this was what happened in the 1970s. Notice that

$$\mathbf{E}_t(\pi_{t+1}) = p_{s_t,0}(\varepsilon_t^u) \mathbf{E}(\pi_{t+1} \mid s_{t+1} = 0, \mathcal{F}_t) + p_{s_t,1}(\varepsilon_t^u) \mathbf{E}(\pi_{t+1} \mid s_{t+1} = 1, \mathcal{F}_t).$$

Rosengren's view highlights that the dynamics of inflation facing adverse supply shocks depend critically on inflation expectation $\mathbf{E}_t(\pi_{t+1})$ through private sector's expectations formation of fu-

¹⁴ A detailed discussion of optimal monetary policy in the presence of markup, or cost push, shock can be found in Clarida et al. (1999).

ture regime, $p_{s_t,j}(\varepsilon_t^u)$, $j = 0, 1$. As demonstrated earlier in Figure 2: the dynamic evolution of private sector's expectation $p_{s_t,j}(\varepsilon_t^u)$ can generate substantial expectations formation effect such that inflations are stabilized, even though monetary policy is under a less aggressive regime. More discussions towards this line can be found in Binder (2017), Coibion, Gorodnichenko and Kamdar (2018), Coibion, Gorodnichenko, Kumar and Pedemonte (2020), among others.

Many questions remain unanswered. For instance, what is the underlying mechanism that the private sector, households and firms, form their expectations of the future regime with respect to adverse supply shocks?¹⁵ While the central bank tries to manage the private sector's inflation expectations by committing to inflation targeting, how would the central bank balance their dual mandate if the inflation does rise like in the 1970s? Nevertheless, my empirical results suggest that it is potentially relevant and essential to research the theoretical mechanism that how adverse supply shock, ε_t^u , affects the private sector's expectations on a future monetary policy stance, $p_{s_t,j}(\varepsilon_t^u)$, $j = 0, 1$. Or, from the central bank's perspective, on which case the adverse markup shocks ε_t^u would induce them to put more weight on stabilizing output and select regime $s_{t+1} = 0$ as a response to these shocks. Answering these theoretical questions is likely a way towards truly endogenizing the U.S. monetary policy regime change and providing micro-foundations to the dynamic expectations formation emphasized in this paper.

5 Conclusions

I introduce a new threshold-switching monetary policy process to a prototype New Keynesian model. This threshold-switching implies a form of endogenous switching with respect to exogenous shocks and induces an expectations feedback mechanism from past fundamental shocks to current monetary policy behavior through agents' expectations on future regime evolution. As an illustration, I solve a Fisherian model of inflation determination embedding the threshold-switching rule analytically. Numerical analysis implies that this expectational channel significantly alters private agents' expectations formation and macroeconomic dynamics. Then, I estimated the threshold-switching New Keynesian model. Several results are in line. First, markup shock has played a predominant role among all exogenous shocks in determining the U.S. monetary policy regime and represented a contribution of 65.6% during the post-WWII history. Second, without markup shocks, eight of the eleven less aggressive regimes would not have occurred. Finally, the empirical results suggest that linking the transmission mechanism of adverse supply shock through the economy with monetary authority's dilemma of dual mandate in such an economic condition could be a meaningful direction to endogenize U.S. monetary policy shifts theoretically.

¹⁵ The ongoing COVID-19 adverse supply shock could be an excellent example of this question. A recent paper Binder (2020) surveyed and studied households' expectations of inflation and unemployment.

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Appendices

A A New Analytical Approach for Expectations Formation

A.1 Analytical Solution to the Fisherian Model

The threshold-switching Fisherian model can be represented as the following system of expectational nonlinear difference equations:

$$i_t = \mathbf{E}_t \pi_{t+1} + r_t, \quad (34)$$

$$r_t = r^* + \sigma_r \varepsilon_t^r, \quad \varepsilon_t^r \sim \mathcal{N}(0, 1), \quad (35)$$

$$i_t = \alpha(s_t) \pi_t, \quad (36)$$

$$s_t = \mathbb{1}\{s_t^* \geq 0\},$$

$$s_{t+1}^* = (1 - \phi)\delta + \phi s_t^* + w_{t+1}, \quad (37)$$

$$\begin{pmatrix} \varepsilon_t^r \\ w_{t+1} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right). \quad (38)$$

Solving these equations is amount to finding a fixed point in a model agent's decision rules. Her information set is $\mathcal{G}_t = \{i_n, \pi_n, r_n, \varepsilon_n^r, s_n\}_{n=0}^t$. Imagining she is given an arbitrary decision rule at time t for determining π_t based on the minimum state variables (ε_t^r, s_t) of the model economy. She will rationally apply this rule to forecast π_{t+1} and evaluate $\mathbf{E}_t \pi_{t+1}$, which, in turn, solves equilibrium π_t . If the resulting inflation π_t , in equilibrium, shares the same relation with (ε_t^r, s_t) as the initial rule, then it is a fixed point. Otherwise, she updates the decision rule accordingly. The procedure is repeated until the decision rule converges to a fixed point.

The guess-and-verify solution method mimics the agent's iterative updating procedure. I start with the following guess of solution form with undetermined coefficients

$$\pi_t = a(s_t, p_{s_t,0}(\varepsilon_t^r))r^* + b(s_t)\varepsilon_t^r. \quad (39)$$

where the first coefficient depends on ε_t^r only through $p_{s_t,0}(\varepsilon_t^r)$.¹⁶ Here, we do not need to include $p_{s_t,1}(\varepsilon_t^r)$ as it is simply $1 - p_{s_t,0}(\varepsilon_t^r)$. Combining (34) and (36) yields the regime-dependent equilibrium relation

$$\alpha(s_t)\pi_t = \mathbf{E}_t \pi_{t+1} + r_t. \quad (40)$$

The conditional expectation $\mathbf{E}_t \pi_{t+1}$ can be evaluated by rewriting solution form (39) one period

¹⁶ This feature is critical for building intuitions. To highlight this feature, I pass the transition probability $p_{s_t,0}(\varepsilon_t^r)$ as an argument to the coefficient $a(s_t, p_{s_t,0}(\varepsilon_t^r))$. But notation-wise, one can replace the it with $a(s_t, \varepsilon_t^r)$ which causes no difference in the derivation below.

forward and plugging in the resulting π_{t+1} . Therefore,

$$\begin{aligned}
\mathbf{E}_t \pi_{t+1} &= \mathbf{E} [a(s_{t+1}, p_{s_{t+1},0}(\varepsilon_{t+1}^r)) \mid \mathcal{G}_t] r^* \\
&= \{p_{s_t,0}(\varepsilon_t^r) \mathbf{E} [a(s_{t+1} = 0, p_{00}(\varepsilon_{t+1}^r)) \mid \mathcal{G}_t] + p_{s_t,1}(\varepsilon_t^r) \mathbf{E} [a(s_{t+1} = 1, p_{10}(\varepsilon_{t+1}^r)) \mid \mathcal{G}_t]\} r^* \\
&= \{p_{s_t,0}(\varepsilon_t^r) \underbrace{\mathbf{E} [a(s_{t+1} = 0, p_{00}(\varepsilon_{t+1}^r))]}_{C_0} + p_{s_t,1}(\varepsilon_t^r) \underbrace{\mathbf{E} [a(s_{t+1} = 1, p_{10}(\varepsilon_{t+1}^r))]}_{C_1}\} r^*, \tag{41}
\end{aligned}$$

where the first equality holds because random variables s_{t+1} and ε_{t+1}^r are independent and shock ε_{t+1}^r has zero mean, hence the second term $\mathbf{E}[b(s_{t+1})\varepsilon_{t+1}^r \mid \mathcal{G}_t]$ in $\mathbf{E}_t \pi_{t+1}$ vanishes. The next equality expands the conditional expectation over the two possible regimes at $t+1$ using transition probabilities $p_{s_t,0}(\varepsilon_t^r)$ and $p_{s_t,1}(\varepsilon_t^r)$ known at t . The last equality is due to the independence between ε_{t+1}^r and $\mathcal{G}_t$. I denote the two expectational terms in the last equality as C_0 and C_1 , which are two undetermined constants since ε_{t+1}^r is i.i.d. and hence the unconditional expectation of any function of ε_{t+1}^r would not depend on time index. Plugging (41) and real rate (35) into (40) and solving π_t yields

$$\pi_t = \frac{p_{s_t,0}(\varepsilon_t^r)C_0 + p_{s_t,1}(\varepsilon_t^r)C_1 + 1}{\alpha(s_t)} r^* + \frac{\sigma_r}{\alpha(s_t)} \varepsilon_t^r. \tag{42}$$

Comparing (42) with (39), I match the unknown coefficients

$$a(s_t, p_{s_t,0}(\varepsilon_t^r)) = \frac{C_0 p_{s_t,0}(\varepsilon_t^r) + C_1 p_{s_t,1}(\varepsilon_t^r) + 1}{\alpha(s_t)}, \tag{43}$$

and

$$b(s_t) = \frac{\sigma_r}{\alpha(s_t)}.$$

To determine $a(s_t, p_{s_t,0}(\varepsilon_t^r))$, I consider $s_t = 0$ and take expectation with respect to ε_t^r on both sides of (43). Notice that the left-hand-side of (43) becomes C_0 because C_0 is a constant, hence time independent. Therefore, I obtain from (43) that

$$C_0 = \frac{\mathbf{E} p_{00}(\varepsilon_t^r)C_0 + \mathbf{E} p_{01}(\varepsilon_t^r)C_1 + 1}{\alpha_0}, \tag{44}$$

where $\alpha_0 = \alpha(s_t = 0)$. Similarly, considering $s_t = 1$ in (43) would yield

$$C_1 = \frac{\mathbf{E} p_{10}(\varepsilon_t^r)C_0 + \mathbf{E} p_{11}(\varepsilon_t^r)C_1 + 1}{\alpha_1}, \tag{45}$$

where $\alpha_1 = \alpha(s_t = 1)$.

From the system of two equations (44) and (45) with two unknown constants C_0, C_1 , I solve for C_0, C_1 , substitute them into (42), organize terms and obtain

$$\pi_t = \frac{1}{\alpha(s_t)} \underbrace{\frac{(\alpha_1 - \alpha_0)p_{s_t,0}(\varepsilon_t^r) + (\alpha_0 - 1)\alpha_1 + \alpha_0 \mathbf{E}p_{10}(\varepsilon_t^r) + \alpha_1 \mathbf{E}p_{01}(\varepsilon_t^r)}{(\alpha_0 - 1)(\alpha_1 - 1) + (\alpha_0 - 1)\mathbf{E}p_{10}(\varepsilon_t^r) + (\alpha_1 - 1)\mathbf{E}p_{01}(\varepsilon_t^r)}}_{a(s_t, p_{s_t,0}(\varepsilon_t^r))} r^* + \underbrace{\frac{\sigma_r}{\alpha(s_t)}}_{b(s_t)} \varepsilon_t^r, \quad (46)$$

where the random coefficient a depends on both s_t and ε_t^r , while b only on s_t . $\mathbf{E}p_{01}(\varepsilon_t^r)$ and $\mathbf{E}p_{10}(\varepsilon_t^r)$ are two constants which need be evaluated. The different representation of solution (10) and (11) can be readily obtained by reorganising terms using (46).

A.2 Numerical Evaluation

The terms $p_{s_t,0}(\varepsilon_t^r)$, $\mathbf{E}p_{01}(\varepsilon_t^r)$ and $\mathbf{E}p_{10}(\varepsilon_t^r)$ in (46) can be evaluated as follows. Given $|\phi| < 1$ and $|\rho| < 1$, combining (37) and (38) yields

$$s_{t+1}^* = (1 - \phi)\delta + \phi s_t^* + \rho \varepsilon_t^r + \sqrt{1 - \rho^2} v_{t+1}, \quad v_{t+1} \sim \mathcal{N}(0, 1).$$

So, the PDF of s_{t+1}^* conditioning on s_t^*, ε_t^r is

$$f(s_{t+1}^* | s_t^*, \varepsilon_t^r) = \varphi(s_{t+1}^*; (1 - \phi)\delta + \phi s_t^* + \rho \varepsilon_t^r, 1 - \rho^2),$$

where $\varphi(\cdot; \mu, \sigma^2)$ is a normal PDF with (μ, σ^2) specified as the mean and variance. Normalize s_{t+1}^* as

$$x = \frac{s_{t+1}^* - (1 - \phi)\delta - \phi s_t^* - \rho \varepsilon_t^r}{\sqrt{1 - \rho^2}}, \quad (47)$$

then $x | s_t^*, \varepsilon_t^r \sim \mathcal{N}(0, 1)$. Notice that the unconditional distribution of regime strength index is $s_t^* \sim \mathcal{N}(\delta, 1/(1 - \phi^2))$. So, we can also normalize s_t^* as $z = \sqrt{1 - \phi^2}(s_t^* - \delta)$, then $z \sim \mathcal{N}(0, 1)$. Substituting s_t^* in (47) with z yields

$$x = \frac{s_{t+1}^* - \delta - \rho \varepsilon_t^r}{\sqrt{1 - \rho^2}} - \frac{\phi z}{\sqrt{(1 - \phi^2)(1 - \rho^2)}},$$

and $x \mid z, \varepsilon_t^r \sim \mathcal{N}(0, 1)$. Then, we can compute ¹⁷

$$\begin{aligned}
p_{00}(\varepsilon_t^r) &= \mathbf{P}(s_{t+1}^* < 0 \mid s_t^* < 0, \varepsilon_t^r) \\
&= \mathbf{P}\left(x < \frac{-\delta - \rho\varepsilon_t^r}{\sqrt{1-\rho^2}} - \frac{\phi z}{\sqrt{(1-\phi^2)(1-\rho^2)}} \mid z < -\delta\sqrt{1-\phi^2}, \varepsilon_t^r\right) \\
&= \frac{\mathbf{P}\left(x < \frac{-\delta - \rho\varepsilon_t^r}{\sqrt{1-\rho^2}} - \frac{\phi z}{\sqrt{(1-\phi^2)(1-\rho^2)}}, z < -\delta\sqrt{1-\phi^2} \mid \varepsilon_t^r\right)}{\mathbf{P}(z < -\delta\sqrt{1-\phi^2})} \\
&= \frac{\int_{-\infty}^{-\delta\sqrt{1-\phi^2}} \Phi\left(\frac{-\delta - \rho\varepsilon_t^r}{\sqrt{1-\rho^2}} - \frac{\phi z}{\sqrt{(1-\phi^2)(1-\rho^2)}}\right) \varphi(z) dz}{\Phi(-\delta\sqrt{1-\phi^2})}. \tag{48}
\end{aligned}$$

Analogously,

$$p_{10}(\varepsilon_t^r) = \frac{\int_{-\delta\sqrt{1-\phi^2}}^{\infty} \Phi\left(\frac{-\delta - \rho\varepsilon_t^r}{\sqrt{1-\rho^2}} - \frac{\phi z}{\sqrt{(1-\phi^2)(1-\rho^2)}}\right) \varphi(z) dz}{\Phi(\delta\sqrt{1-\phi^2})}. \tag{49}$$

Then, it follows from (48) and (49) that $p_{01}(\varepsilon_t^r) = 1 - p_{00}(\varepsilon_t^r)$ and $p_{11}(\varepsilon_t^r) = 1 - p_{10}(\varepsilon_t^r)$.

Using $p_{00}(\varepsilon_t^r)$ as an example, Figure 6 shows how the parameters (ϕ, δ, ρ) affect the TVTP. When $\rho = 0$, transition probability $p_{00}(\varepsilon_t^r)$ given in (48) becomes a constant, which corresponds to the exogenous-switching case. Thus, it is clear that (ϕ, δ) determine the magnitude of transition probability while ρ controls its fluctuation.

Next, let $J = \phi/\sqrt{(1-\phi^2)(1-\rho^2)}$, I rewrite the numerators in (48) and (49) as bivariate normal CDF's and obtain

$$p_{00}(\varepsilon_t^r) = \frac{\int_{-\infty}^{-\delta\sqrt{1-\phi^2}} \int_{-\infty}^{\frac{-\delta - \rho\varepsilon_t^r}{\sqrt{1-\rho^2}}} \varphi_1(z, y) dy dz}{\Phi(-\delta\sqrt{1-\phi^2})}, \quad \varphi_1(z, y) \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & J \\ J & 1+J^2 \end{pmatrix}\right),$$

and,

$$p_{10}(\varepsilon_t^r) = \frac{\int_{-\infty}^{\delta\sqrt{1-\phi^2}} \int_{-\infty}^{\frac{-\delta - \rho\varepsilon_t^r}{\sqrt{1-\rho^2}}} \varphi_2(z, y) dy dz}{\Phi(\delta\sqrt{1-\phi^2})}, \quad \varphi_2(z, y) \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & -J \\ -J & 1+J^2 \end{pmatrix}\right).$$

¹⁷ A more sophisticated agent would use all historical real rate shocks rather than just the one-period lag shock to forecast the next period's regime probability. In that case, one would compute $p_{00}(\varepsilon_t^r, \varepsilon_{t-1}^r, \dots) = \mathbf{P}(s_{t+1}^* < 0 \mid s_t^* < 0, \varepsilon_t^r, \varepsilon_{t-1}^r, \dots)$. In other words, I have used an approximate transition probability in deriving the solution. See footnote 7 for a related remark.

Figure 6. Time-Varying Transition Probabilites from Regime 0 to Regime 0

Note: The blue curve depicts the TVTP of staying at regime-0 with respect to the real rate shock in the x-axis. Parameters are set at $(\phi, \delta, \rho) = (0.7, 0, 0.9)$. The red horizontal line shows the corresponding constant transition probability with parameters $(\phi, \delta, \rho) = (0.7, 0, 0)$.

Similarly, let $K = 1/((1 - \phi^2)(1 - \rho^2))$ and $M = \rho/\sqrt{1 - \rho^2}$, I specify the numerators in $\mathbf{E} p_{00}(\varepsilon_t^r)$ and $\mathbf{E} p_{10}(\varepsilon_t^r)$ as trivariate normal CDF's and obtain

$$\begin{aligned} \mathbf{E} p_{00}(\varepsilon_t^r) &= \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{-\delta\sqrt{1-\phi^2}} \Phi\left(\frac{-\delta - \rho\varepsilon_t^r}{\sqrt{1-\rho^2}} - \frac{\phi z}{\sqrt{(1-\phi^2)(1-\rho^2)}}\right) \varphi(z) \varphi(\varepsilon_t^r) dz d\varepsilon_t^r}{\Phi(-\delta\sqrt{1-\phi^2})} \\ &= \frac{\int_{-\infty}^{-\delta\sqrt{1-\phi^2}} \int_{-\infty}^{-\delta/\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \varphi_3(z, y, \varepsilon) d\varepsilon dy dz}{\Phi(-\delta\sqrt{1-\phi^2})}, \end{aligned}$$

with

$$\varphi_3(z, y, \varepsilon) \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & J & 0 \\ J & K & M \\ 0 & M & 1 \end{pmatrix}\right),$$

and,

$$\begin{aligned} \mathbf{E} p_{10}(\varepsilon_t^r) &= \frac{\int_{-\infty}^{\infty} \int_{-\delta\sqrt{1-\phi^2}}^{\infty} \Phi\left(\frac{-\delta - \rho\varepsilon_t^r}{\sqrt{1-\rho^2}} - \frac{\phi z}{\sqrt{(1-\phi^2)(1-\rho^2)}}\right) \varphi(z) \varphi(\varepsilon_t^r) dz d\varepsilon_t^r}{\Phi(\delta\sqrt{1-\phi^2})} \\ &= \frac{\int_{-\infty}^{\delta\sqrt{1-\phi^2}} \int_{-\infty}^{-\delta/\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \varphi_4(z, y, \varepsilon) d\varepsilon dy dz}{\Phi(\delta\sqrt{1-\phi^2})}, \end{aligned}$$

with

$$\varphi_4(z, y, \varepsilon) \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & -J & 0 \\ -J & K & M \\ 0 & M & 1 \end{pmatrix}\right).$$

Based on these derivations, we are able to evaluate the exact solution (46) using multivariate normal CDF in numerical software.

B Equilibrium Relations and Steady States

The dynamic IS equation and the NKPC derived from the household and firms' optimizing behaviors, regarding the detrended variables, are

$$1 = \mathbf{E}_t \left[\left(\frac{c_t}{c_{t+1}} \right)^\epsilon \frac{d_{t+1}}{d_t} \frac{\beta R_t}{e^\lambda a_{t+1} \Pi_{t+1}} \right], \quad (50)$$

$$\begin{aligned} 1 = & u_t c_t^\epsilon - \psi \left(\frac{\Pi_t}{\Pi_{s_t}^*} - 1 \right) \left[\left(\frac{u_t}{2} - 1 \right) \frac{\Pi_t}{\Pi_{s_t}^*} + \frac{u_t}{2} \right] \\ & + \beta \psi (u_t - 1) \mathbf{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\epsilon} \frac{d_{t+1}}{d_t} \frac{y_{t+1}}{y_t} \left(\frac{\Pi_{t+1}}{\Pi_{s_{t+1}}^*} - 1 \right) \frac{\Pi_{t+1}}{\Pi_{s_{t+1}}^*} \right]. \end{aligned} \quad (51)$$

These two equations, along with market clearing condition (17), threshold-switching monetary policy (18)-(20), and exogenous shocks (21)-(26), form the full characterization of the threshold-switching New Keynesian model.

When shocks are shut down and inflation is set at target level, (50) implies that

$$1 = \bar{p}_{s_t,0} \left(\frac{\beta}{e^\lambda} \frac{R_{s_t}^*}{\Pi_0^*} \right) + \bar{p}_{s_t,1} \left(\frac{\beta}{e^\lambda} \frac{R_{s_t}^*}{\Pi_1^*} \right),$$

where $\bar{p}_{s_t,0}$ and $\bar{p}_{s_t,1}$ are the transition probabilities from regime s_t to regime 0 and 1 when shocks are turned off. The presence of transition probabilities in the expression indicates that regime switches occur in steady state. So, steady-state interest rate is also regime dependent and is solved

as

$$R_{st}^* = \frac{e^\lambda}{\beta} \left(\frac{\bar{p}_{st,0}}{\Pi_0^*} + \frac{\bar{p}_{st,1}}{\Pi_1^*} \right)^{-1}.$$

Finally, from (51) and (17), absent nominal rigidities ($\psi = 0$) and markup shock, it is straightforward to calculate the aggregate output as

$$Y_t^* = \left(\frac{\theta - 1}{\theta} \right)^{1/\epsilon} g_t A_t,$$

which is the output target level of the central bank.

C Algorithm for Threshold-Switching Kalman Filter

C.1 Prerequisites

The prerequisites to developing the filter are a few properties of normal distribution, mixture distribution, and truncated normal distribution, which I collect below.

Property 1 (Marginal Distribution). Let $u \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $v \sim \mathcal{N}(u, \sigma_2^2)$ where the realized value of u is given as the mean of v . Then, we have the following result:

$$\int_u \varphi(v; u, \sigma_2^2) \varphi(u; \mu_1, \sigma_1^2) du = \varphi(v; \mu_1, \sigma_1^2 + \sigma_2^2).$$

Property 2 (Sum of Independent Normal Distributions). Let u, v be independent normal distributions, then their sum is also normally distributed. That is, if $u \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $v \sim \mathcal{N}(\mu_2, \sigma_2^2)$ are independent, and $z = u + v$, then $z \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Property 3 (Mixture Distribution). Let U, V denote multivariate random variables from distribution $\varphi(u; \mu_1, \Sigma_1^2)$ and $\varphi(v; \mu_2, \Sigma_2^2)$, where μ_i, Σ_i^2 are the corresponding mean and variance. Given weights w_1, w_2 such that $w_1 + w_2 = 1$ and $w_i \geq 0$. Let Z denote the random variable for the combined density $w_1 \varphi(u; \mu_1, \Sigma_1^2) + w_2 \varphi(v; \mu_2, \Sigma_2^2)$, which represents a mixture of distributions U and V . Then,

$$\begin{aligned} \mathbf{E}(Z) &= w_1 \mu_1 + w_2 \mu_2, \\ \mathbf{Var}(Z) &= \sum_{i=0}^1 w_i \{ \Sigma_i^2 + [\mathbf{E}(Z) - \mu_i][\mathbf{E}(Z) - \mu_i]' \}. \end{aligned}$$

Property 4 (Right-Truncated Normal Distribution with Truncation Point Zero). Let x has a normal density $\varphi(\mu, \sigma^2)$ and lies within the interval $(-\infty, 0)$. Then x conditional on $x < 0$

follows a truncated normal distribution. Its PDF, denoted as φ^- , is

$$\varphi^-(x; \mu, \sigma^2) = \begin{cases} \frac{\varphi(x; \mu, \sigma^2)}{\mathbf{P}(x < 0)}, & \text{if } x \leq 0. \\ 0, & \text{otherwise.} \end{cases}$$

Let x^- denotes the random variable of the right truncated distribution, and let $\tau = -\mu/\sigma$. Then, its mean and variance is given as

$$\mathbf{E}(x^-) = \mu - \sigma\varphi(\tau)/\Phi(\tau),$$

$$\mathbf{Var}(x^-) = \sigma^2 \{1 - \tau\varphi(\tau)/\Phi(\tau) - [\varphi(\tau)/\Phi(\tau)]^2\}.$$

Property 5 (Left-Truncated Normal Distribution with Truncation Point Zero). Let x has a normal density $\varphi(\mu, \sigma^2)$ and lies within the interval $(0, \infty)$. Then x conditional on $x > 0$ follows a truncated normal distribution. Its PDF, denoted as φ^+ , is

$$\varphi^+(x; \mu, \sigma^2) = \begin{cases} \frac{\varphi(x; \mu, \sigma^2)}{\mathbf{P}(x > 0)}, & \text{if } x \geq 0. \\ 0, & \text{otherwise.} \end{cases}$$

Let x^+ denotes the random variable of the left truncated distribution, and let $\tau = -\mu/\sigma$. Then, its mean and variance is given as

$$\mathbf{E}(x^+) = \mu + \sigma\varphi(\tau)/\Phi(-\tau),$$

$$\mathbf{Var}(x^+) = \sigma^2 \{1 + \tau\varphi(\tau)/\Phi(-\tau) - [\varphi(\tau)/\Phi(-\tau)]^2\}.$$

C.2 Filtering Algorithm

Consider the threshold-switching state-space representation:

$$\begin{aligned} y_t &= D(s_t) + Z(s_t)x_t + \Sigma_\nu(s_t)^{1/2}\nu_t, \quad \nu_t \sim \mathcal{N}(0, I_{n_\nu}), \\ x_t &= C(s_t) + M(s_t)x_{t-1} + G(s_t)\varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, I_{n_\varepsilon}), \end{aligned}$$

where y_t is the vector of observables, ν_t is measurement error vector, x_t is the state vector, ε_t is the transition innovations vector, and matrices $D, Z, \Sigma_\nu, C, M, G$ are allowed to switch between two regimes according to $s_t = 0$ or 1 .¹⁸ The regime indicator $s_t = \mathbb{1}\{s_t^* \geq 0\}$ is determined by the

¹⁸ It is straightforward to include predetermined variables with switching coefficients on both measurement and transition equations. I omit them for simplicity.

autoregressive latent regime strength index s_t^* following

$$s_{t+1}^* = (1 - \phi)\delta + \phi s_t^* + w_{t+1}. \quad (52)$$

I introduce the expectations feedback by assuming

$$\begin{pmatrix} \varepsilon_t \\ w_{t+1} \end{pmatrix} \sim \mathcal{N} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} I_{n_\varepsilon} & \rho' \\ \rho & 1 \end{pmatrix} \right). \quad (53)$$

Equations (52) and (53) imply that

$$s_{t+1}^* = (1 - \phi)\delta + \phi s_t^* + \rho' \varepsilon_t + \sqrt{1 - \rho' \rho} v_{t+1}, \quad v_{t+1} \sim \mathcal{N}(0, 1),$$

where v_{t+1} is orthogonal to ε_t . To facilitate estimation, I augment ε_t into x_t and denote $\varsigma_t = [x_t', \varepsilon_t']'$. For notational convenience, define $\eta_t = \phi s_t^* + \rho' \varepsilon_t$. Also, define the conditional means and variances, for $i = 0, 1$, as

$$\begin{aligned} \mu_{t|t-1}^s &= \mathbf{E}(s_t^* \mid \mathcal{F}_{t-1}), & \Omega_{t|t-1}^s &= \mathbf{Var}(s_t^* \mid \mathcal{F}_{t-1}), \\ \mu_{t|t}^s &= \mathbf{E}(s_t^* \mid \mathcal{F}_t), & \Omega_{t|t}^s &= \mathbf{Var}(s_t^* \mid \mathcal{F}_t), \\ \varsigma_{t|t-1}^i &= \mathbf{E}(\varsigma_t \mid s_t = i, \mathcal{F}_{t-1}), & Q_{t|t-1}^i &= \mathbf{Var}(\varsigma_t \mid s_t = i, \mathcal{F}_{t-1}), \\ \varsigma_{t|t}^i &= \mathbf{E}(\varsigma_t \mid s_t = i, \mathcal{F}_t), & Q_{t|t}^i &= \mathbf{Var}(\varsigma_t \mid s_t = i, \mathcal{F}_t), \\ y_{t|t-1}^i &= \mathbf{E}(y_t \mid s_t = i, \mathcal{F}_{t-1}), & H_{t|t-1}^i &= \mathbf{Var}(y_t \mid s_t = i, \mathcal{F}_{t-1}), \end{aligned}$$

as well as the probabilities

$$p_{t|t-1}^i = \mathbf{P}(s_t = i \mid \mathcal{F}_{t-1}), \quad p_{t|t}^i = \mathbf{P}(s_t = i \mid \mathcal{F}_t),$$

where $\mathcal{F}_t = \{y_n\}_{n=1}^t$.

1. Initialization. Initialize $(\mu_{0|0}^s, \Omega_{0|0}^s)$ using the invariant distribution of s_t^* , i.e., $s_{0|0}^* \sim \mathcal{N}(\delta, 1/(1 - \phi^2))$. So, it implies

$$p_{0|0}^0 = \Phi(-\delta\sqrt{1 - \phi^2}), \quad p_{0|0}^1 = 1 - p_{0|0}^0.$$

Also, initialize $(\varsigma_{0|0}^i, Q_{0|0}^i)$ using invariant distributions of ς_t under $i = 0, 1$. Then, I collapse the

mixture normal into

$$\varsigma_{0|0} = \sum_{i=0}^1 p_{0|0}^i \varsigma_{0|0}^i,$$

$$Q_{0|0} = \sum_{i=0}^1 p_{0|0}^i \left[Q_{0|0}^i + (\varsigma_{0|0} - \varsigma_{0|0}^i)(\varsigma_{0|0} - \varsigma_{0|0}^i)' \right].$$

2. Recursion. Go through the following steps for $t = 1, \dots, T$.

2.1. Prediction. First, I forecast density of s_t^* as

$$\begin{aligned} f(s_t^* | \mathcal{F}_{t-1}) &= \int f(s_t^* | \eta_{t-1}, \mathcal{F}_{t-1}) f(\eta_{t-1} | \mathcal{F}_{t-1}) d\eta_{t-1} \\ &= \int f(s_t^* | \eta_{t-1}) f(\eta_{t-1} | \mathcal{F}_{t-1}) d\eta_{t-1} \\ &\approx \int \varphi(s_t^*; (1-\phi)\delta + \eta_{t-1}, 1-\rho'\rho) \varphi(\eta_{t-1}; \underbrace{\phi\mu_{t-1|t-1}^s + \rho'\varsigma_{t-1|t-1}^\varepsilon}_{\mu_{t|t-1}^s}, \underbrace{\phi^2\Omega_{t-1|t-1}^s + \rho'Q_{t-1|t-1}^\varepsilon}_{\Omega_{t|t-1}^s}) d\eta_{t-1} \\ &= \varphi\left(s_t^*; \underbrace{(1-\phi)\delta + \phi\mu_{t-1|t-1}^s + \rho'\varsigma_{t-1|t-1}^\varepsilon}_{\mu_{t|t-1}^s}, \underbrace{1-\rho'\rho + \phi^2\Omega_{t-1|t-1}^s + \rho'Q_{t-1|t-1}^\varepsilon}_{\Omega_{t|t-1}^s}\right), \end{aligned} \quad (54)$$

where the second equality is due to the conditional independence of s_t^* and $\mathcal{F}_{t-1}$ conditioning on η_{t-1} , and the last equality uses Property 1 in the Prerequisites. In the approximate equality, the first density is exact. For the second density, notice that $\eta_{t-1} = \phi s_{t-1}^* + \rho' \varepsilon_{t-1}$. I approximated

$$f(s_{t-1}^* | \mathcal{F}_{t-1}) \approx \varphi(s_{t-1}^*; \mu_{t-1|t-1}^s, \Omega_{t-1|t-1}^s),$$

where the mean $\mu_{t-1|t-1}^s$ and variance $\Omega_{t-1|t-1}^s$ are from initialization or last updating step. Similarly, I approximate

$$f(\varepsilon_{t-1} | \mathcal{F}_{t-1}) \approx \varphi(\varepsilon_{t-1}; \varsigma_{t-1|t-1}^\varepsilon, Q_{t-1|t-1}^\varepsilon),$$

where $\varsigma_{t-1|t-1}^\varepsilon, Q_{t-1|t-1}^\varepsilon$ are subvector and submatrix of $\varsigma_{t-1|t-1}, Q_{t-1|t-1}$ according to indices of ε in state vector ς . Then, the second density can be obtained by Property 2 due to independence of s_{t-1}^* and ε_{t-1} .¹⁹ From (54), the regime probabilities can be forecasted as

$$p_{t|t-1}^0 = \mathbf{P}(s_t^* < 0 | \mathcal{F}_{t-1}) = \Phi\left(\frac{\delta - \mu_{t|t-1}^s}{\sqrt{\Omega_{t|t-1}^s}}\right)$$

¹⁹ For $t \geq 1$, the true data generating process (DGP) of $f(\varsigma_{t-1} | \mathcal{F}_{t-1})$ is a 2^t -folds mixture of normal distributions. $f(s_{t-1}^* | \mathcal{F}_{t-1})$ is initialized as a normal density for $t = 1$. For $t > 1$, the DGP of $f(s_{t-1}^* | \mathcal{F}_{t-1})$ would be complex combinations of normal, mixture of normal, and mixture of truncated normal distributions. The part of truncated normal distributions enters in the updating step for s_t^* .

and $p_{t|t-1}^1 = 1 - p_{t|t-1}^0$. Next, for $i = 0, 1$, we have

$$\begin{aligned}\varsigma_{t|t-1}^i &= C(i) + M(i)\varsigma_{t-1|t-1}, \\ Q_{t|t-1}^i &= M(i)Q_{t-1|t-1}M(i)' + G(i)G(i)',\end{aligned}$$

for the state vector, where $(\varsigma_{t-1|t-1}, Q_{t-1|t-1})$ are obtained from the previous updating step or initialization.

2.2. Likelihood. For $i = 0, 1$, forecast the observables as

$$\begin{aligned}y_{t|t-1}^i &= D(i) + Z(i)\varsigma_{t|t-1}^i, \\ H_{t|t-1}^i &= Z(i)Q_{t|t-1}^iZ(i)' + \Sigma_\nu.\end{aligned}$$

So, the likelihood for period t is

$$f(y_t | \mathcal{F}_{t-1}) = \sum_{i=0}^1 p_{t|t-1}^i \varphi(y_t; y_{t|t-1}^i, H_{t|t-1}^i).$$

2.3. Updating. To update, use the Bayes' formula for $s_t = 0$ to get

$$p_{t|t}^0 = \frac{p_{t|t-1}^0 \varphi(y_t; y_{t|t-1}^0, H_{t|t-1}^0)}{f(y_t | \mathcal{F}_{t-1})}. \quad (55)$$

Then, for $s_t = 1$, we have $p_{t|t}^1 = 1 - p_{t|t}^0$. Use the Bayes' formula again to obtain

$$\begin{aligned}f(s_t^* | \mathcal{F}_t) &= \frac{f(y_t | s_t^*, \mathcal{F}_{t-1})f(s_t^* | \mathcal{F}_{t-1})}{f(y_t | \mathcal{F}_{t-1})} \\ &= \frac{f(y_t | s_t, \mathcal{F}_{t-1})f(s_t^* | \mathcal{F}_{t-1})}{f(y_t | \mathcal{F}_{t-1})} \\ &= \frac{\varphi(y_t; y_{t|t-1}^0, H_{t|t-1}^0)f(s_t^* | \mathcal{F}_{t-1})}{f(y_t | \mathcal{F}_{t-1})} \mathbb{1}\{s_t^* < 0\} + \frac{\varphi(y_t; y_{t|t-1}^1, H_{t|t-1}^1)f(s_t^* | \mathcal{F}_{t-1})}{f(y_t | \mathcal{F}_{t-1})} \mathbb{1}\{s_t^* \geq 0\} \\ &= p_{t|t}^0 \frac{f(s_t^* | \mathcal{F}_{t-1}) \mathbb{1}\{s_t^* < 0\}}{p_{t|t-1}^0} + p_{t|t}^1 \frac{f(s_t^* | \mathcal{F}_{t-1}) \mathbb{1}\{s_t^* \geq 0\}}{p_{t|t-1}^1}.\end{aligned}$$

It implies that $f(s_t^* | \mathcal{F}_t)$ is a mixture of two truncated normal densities with weights $p_{t|t}^0$ and $p_{t|t}^1$, provided that $f(s_t^* | \mathcal{F}_{t-1})$ is normal. Let $\tau_t = -\mu_{t|t-1}^s / \sqrt{\Omega_{t|t-1}^s}$. Using Properties 3, 4, and 5, I

obtain the mean $\mu_{t|t}^s$ and variance $\Omega_{t|t}^s$ for density $f(s_t^* | \mathcal{F}_t)$ as below:

$$\begin{aligned}
\mu_\tau^0 &= \mu_{t|t-1}^s - \sqrt{\Omega_{t|t-1}^s} \phi(\tau_t) / \Phi(\tau_t), \\
\Omega_\tau^0 &= \Omega_{t|t-1}^s \{1 - \tau_t \phi(\tau_t) / \Phi(\tau_t) - [\phi(\tau_t) / \Phi(\tau_t)]^2\}, \\
\mu_\tau^1 &= \mu_{t|t-1}^s + \sqrt{\Omega_{t|t-1}^s} \phi(\tau_t) / \Phi(-\tau_t), \\
\Omega_\tau^1 &= \Omega_{t|t-1}^s \{1 + \tau_t \phi(\tau_t) / \Phi(-\tau_t) - [\phi(\tau_t) / \Phi(-\tau_t)]^2\}, \\
\mu_{t|t}^s &= \sum_{i=0}^1 p_{t|t}^i \mu_\tau^i, \\
\Omega_{t|t}^s &= \sum_{i=0}^1 p_{t|t}^i [\Omega_\tau^i + (\mu_{t|t}^s - \mu_\tau^i)^2],
\end{aligned}$$

where $(\mu_\tau^0, \Omega_\tau^0)$ and $(\mu_\tau^1, \Omega_\tau^1)$ are the mean and variance for the right-truncated and left-truncated normal densities, respectively. Next, the Kalman filter implies

$$\begin{aligned}
\varsigma_{t|t}^i &= \varsigma_{t|t-1}^i + Q_{t|t-1}^i Z(i)' (H_{t|t-1}^i)^{-1} (y_t - y_{t|t-1}^i), \\
Q_{t|t}^i &= Q_{t|t-1}^i - Q_{t|t-1}^i Z(i)' (H_{t|t-1}^i)^{-1} Z(i) Q_{t|t-1}^i.
\end{aligned}$$

Then, the filtered states are

$$\begin{aligned}
\varsigma_{t|t} &= \sum_{i=0}^1 p_{t|t}^i \varsigma_{t|t}^i, \\
Q_{t|t} &= \sum_{i=0}^1 p_{t|t}^i \left[Q_{t|t}^i + (\varsigma_{t|t} - \varsigma_{t|t}^i)(\varsigma_{t|t} - \varsigma_{t|t}^i)' \right].
\end{aligned}$$

3. Aggregation. After the iteration, aggregating the period-likelihood to obtain

$$\ln p(Y_{1:T}) = \ln f(y_1) + \sum_{t=2}^T \ln f(y_t | \mathcal{F}_{t-1}).$$

This filter can extract the regime index innovation $v_{t|t}$ as a by-product as below. From $s_t^* = (1 - \phi)\delta + \eta_{t-1} + \sqrt{1 - \rho' \rho} v_t$, we have

$$\mathbf{E}(v_t | \mathcal{F}_t) = [\mu_{t|t}^s - (1 - \phi)\delta - \mathbf{E}(\eta_{t-1} | \mathcal{F}_t)] / \sqrt{1 - \rho' \rho}, \quad (56)$$

where $\mu_{t|t}^s$ has already been extracted. To obtain $\mathbf{E}(\eta_{t-1} | \mathcal{F}_t)$, I evaluate the density of η_{t-1}

conditional on $\mathcal{F}_t$ as below:

$$\begin{aligned}
f(\eta_{t-1} | \mathcal{F}_t) &= \int f(\eta_{t-1}, s_t^* | \mathcal{F}_t) ds_t^* \\
&= \int \frac{f(y_t | \eta_{t-1}, s_t^*, \mathcal{F}_{t-1}) f(s_t^* | \eta_{t-1}, \mathcal{F}_{t-1}) f(\eta_{t-1} | \mathcal{F}_{t-1})}{f(y_t | \mathcal{F}_{t-1})} ds_t^* \\
&\approx \int \frac{f(y_t | s_t^*, \mathcal{F}_{t-1}) f(s_t^* | \eta_{t-1}, \mathcal{F}_{t-1}) f(\eta_{t-1} | \mathcal{F}_{t-1})}{f(y_t | \mathcal{F}_{t-1})} ds_t^* \\
&= \int \frac{f(y_t | s_t, \mathcal{F}_{t-1}) \varphi(s_t^*; (1-\phi)\delta + \eta_{t-1}, 1 - \rho'\rho) \varphi(\eta_{t-1}; \mu_{t-1|t-1}^\eta, \Omega_{t-1|t-1}^\eta)}{f(y_t | \mathcal{F}_{t-1})} ds_t^* \\
&= \varphi(\eta_{t-1}; \mu_{t-1|t-1}^\eta, \Omega_{t-1|t-1}^\eta) \left\{ \frac{\varphi(y_t; y_{t|t-1}^0, H_{t|t-1}^0)}{f(y_t | \mathcal{F}_{t-1})} \int_{-\infty}^0 \varphi(s_t^*; (1-\phi)\delta + \eta_{t-1}, 1 - \rho'\rho) ds_t^* \right. \\
&\quad \left. + \frac{\varphi(y_t; y_{t|t-1}^1, H_{t|t-1}^1)}{f(y_t | \mathcal{F}_{t-1})} \int_0^\infty \varphi(s_t^*; (1-\phi)\delta + \eta_{t-1}, 1 - \rho'\rho) ds_t^* \right\} \\
&= \varphi(\eta_{t-1}; \mu_{t-1|t-1}^\eta, \Omega_{t-1|t-1}^\eta) \left\{ \frac{p_{t|t}^0}{p_{t|t-1}^0} \Phi \left[\frac{(\phi-1)\delta - \eta_{t-1}}{\sqrt{1-\rho'\rho}} \right] + \frac{p_{t|t}^1}{p_{t|t-1}^1} \Phi \left[\frac{(1-\phi)\delta + \eta_{t-1}}{\sqrt{1-\rho'\rho}} \right] \right\},
\end{aligned}$$

where the third approximate equality occurs since η_{t-1} contains information about y_t conditional on s_t^* and $\mathcal{F}_{t-1}$,²⁰ the fourth equality holds due to the independence of y_t and s_t^* conditional on $s_t, \mathcal{F}_{t-1}$ in the first density and the independence of s_t^* and $\mathcal{F}_{t-1}$ conditional on η_{t-1} in the second density of the numerator. Also in this equality, the last density in the numerator has been obtained from (54) and I have denoted $(\mu_{t-1|t-1}^\eta, \Omega_{t-1|t-1}^\eta) = (\phi\mu_{t-1|t-1}^s + \rho'\varsigma_{t-1|t-1}^\varepsilon, \phi^2\Omega_{t-1|t-1}^s + \rho'Q_{t-1|t-1}^\varepsilon\rho)$. In the last equality, I used result from the updating step (55) for $p_{t|t}^0$, and the corresponding formula for $p_{t|t}^1$. Then, we can numerically compute

$$\mathbf{E}(\eta_{t-1} | \mathcal{F}_t) = \int \eta_{t-1} f(\eta_{t-1} | \mathcal{F}_t) d\eta_{t-1}.$$

Therefore, $\mathbf{E}(v_t | \mathcal{F}_t)$ can be readily extracted using (56). Note that this numerical integration needs not to be done in the main filtering, so computational efficiency of estimation can be guaranteed. A final remark of the filter is that it is straightforward to extend it to 2^l folds, rather than 2 folds in the current version, by carrying l lagged periods without immediate collapsing. As l increases, accuracy can be improved. Readers are referred to Kim (1994) for details regarding this extension associated with the author's exogenous switching Kalman filter.

²⁰ Precisely, η_{t-1} is correlated with ε_{t-1} , which affects ς_t hence y_t . However, compared to the other channel that η_{t-1} directly affects s_t^* , then s_t and y_t , the omitted channel is expected to contain relatively much less information about y_t . The accuracy of estimated regime index shock in the simulation experiment below validates this approximation.

C.3 Simulation

This subsection examines the performance of the approximated threshold-switching Kalman filter by comparing results to those from a textbook bootstrap particle filter. The results are striking: the approximation employed in the filter performs an excellent job, with a considerable advantage in computation time.

I consider a toy state-space model for simulation. The model writes

$$y_t = (1, 1) \begin{pmatrix} x_t \\ z_t \end{pmatrix} + \sigma_\nu \nu_t, \quad \nu_t \sim \mathcal{N}(0, 1),$$

$$\begin{pmatrix} x_t \\ z_t \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12}(s_t) \\ 0 & g_{22}(s_t) \end{pmatrix} \begin{pmatrix} x_{t-1} \\ z_{t-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \sigma_\varepsilon \end{pmatrix} \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, 1),$$

where s_t switches between state 0 and 1 according to the same threshold switching rule as before:

$$s_t = \mathbb{1}\{s_t^* \geq 0\},$$

$$s_{t+1}^* = (1 - \phi)\delta + \phi s_t^* + w_{t+1},$$

and

$$\begin{pmatrix} \varepsilon_t \\ w_{t+1} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right).$$

The realized structural shock ε_t at t travels to z_t , and in the next period, to z_{t+1} and x_{t+1} , then towards the future through the transition system. Its propagation is subject to the switching autoregressive coefficient $g_{22}(s_t)$ of z_t and the sensitivity coefficient $g_{12}(s_t)$ of x_t to z_{t-1} . Meanwhile, the threshold switching creates a new channel that ε_t would impact s_{t+1}^* , therefore entering the next period's transition system through s_{t+1} . Moreover, s_{t+1}^* carries ε_t and generates cumulative impact on all future regime strength index, regime indicators, and hence the future state vectors.

I use a textbook bootstrap particle filter as a benchmark for comparison. Readers are referred to Section 8.1 in Herbst and Schorfheide (2015) for a description of a generic algorithm. It is straightforward to adapt it to the threshold-switching state-space model in question. In essence, the particle filter draws an initial swarm of particles $(s_{0|0}^*, (x_{0|0}, z_{0|0}), \varepsilon_{0|0})$ from their invariant distributions, propagates these particles through the transition system and regime index, and evaluates the likelihood function. It is worthwhile to mention that the invariant distribution for the state vector $(x_{0|0}, z_{0|0})$ is a mixture of two normal distributions with $p_{0|0}^0$ and $p_{0|0}^1$ as the weights, calculated from particles of $s_{0|0}^*$. Also, I employed the systematic resampling scheme in the resampling step.

For simulation, I set parameters at $g_{11} = 0.7$, $(g_{12}(0), g_{12}(1)) = (0.8, 0.2)$, $(g_{22}(0), g_{22}(1)) =$

Figure 7. Comparison of Filters: Likelihood Surface

Note: In each subplot, the red solid line depicts the likelihood curve evaluated by the modified Kalman filter, while the blue dashed line displays that from the bootstrap particle filter.

$(0.1, 0.9)$, $(\phi, \delta, \rho) = (0.7, 0.5, 0.9)$, and $(\sigma_\varepsilon, \sigma_\nu) = (0.5, 0.2)$, and then generate 100-period observables according to the model. For a preassigned grid of parameters around the true values, Figure 7 depicts the likelihood surface evaluated by the modified Kalman filter and bootstrap particle filter. Since the likelihood surface by particle filter is stochastic, I draw 200,000 particles, which is more than enough for the small state-space model, to improve the numerical stability and smoothness of the estimated likelihood surface. We can see that, for most of the parameter values, the likelihood values evaluated by the two filters are quite close, with differences smaller than 0.61, except for the left-end of $g_{12}(0)$ and σ_ε , which are away from their true values. Further, Figure 8 compares the filtered variables and period log-likelihood from the two filters, based on the true parameter values. All series are almost identical. Only $s_{t|t}^*$ and $v_{t|t}$ exhibit some tiny discrepancies with a magnitude

Figure 8. Comparison of Filters: Filtered Variables and Period Log-Likelihood

Note: The first five subplots print the filtered variables from the bootstrap particle filter in dashed blue lines and that from the modified Kalman filter in solid red lines. The period log-likelihood from the two filters are reported in the last subplot.

of smaller than 0.43 for $s_{t|t}^*$ and smaller than 0.033 for $v_{t|t}$. Moreover, the computation time of the modified Kalman filter is about 64 times (10.0 vs. 650.9 seconds) faster than the particle filter. Even if we reduce the number of particles to 100,000, it is still about 29 times (10.0 vs. 302.6 seconds) faster. For a DSGE model, the computational advantage of the analytic modified Kalman filter against the simulation-based particle filter can be even more substantial. All these results demonstrate that the modified Kalman filter manages to deliver a great performance.

D Figures for MCMC Convergence

Figure 9. Diagnostics of MCMC: Recursive Means of Posterior Draws

Note: For each subplot in the figure, it reports the recursive mean of 400,000 MCMC posterior draws for the corresponding parameter of the DSGE model.

Figure 10. Diagnostics of MCMC: Trace Plots of Posterior Draws

Note: For each subplot in the figure, it reports the trace plot of 400,000 MCMC posterior draws for the corresponding parameter of the DSGE model.

Figure 11. Diagnostics of MCMC: Autocorrelation Functions of Posterior Draws

Note: For each subplot in the figure, it reports the autocorrelation function of 400,000 MCMC posterior draws for the corresponding parameter of the DSGE model. The lags, as shown in each x-axis, range from 0 to 1000.