Endogenous Bank Networks and Contagion

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Endogenous Bank Networks and Contagion∗

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Abstract

I develop a model to study two related questions: how bank decisions to form connections depend on fundamentals; and how financial stability depends on bank network structure. In my model, banks are connected through two layers of networks: interbank debts and banks’ common investments in non-financial firms. These layers of interconnections are incentivized by diversified investments when banks maximize their expected equity values according to mean-variance rules. Comparative statics of a small number of banks indicates that, in equilibrium, as banks become less risk averse, they tend to issue more debts and form more links within the banking sector. Furthermore, I conduct numerical computations for bank default probabilities in a circle network and a more connected network. The results demonstrate that increased bank interconnectedness and common asset holdings significantly reduce systemic stability.

JEL Classification: D85, G20, G11

Keywords: Endogenous Network, Financial Contagion, Interbank Debt, Portfolio Selection

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1 Introduction

Interbank debts and banks’ common asset holdings of non-financial firms could provide channels of risk propagation in the financial sector, and this could potentially lead to systemic vulnerability. In this paper, I propose a model of an endogenous bank network adopting portfolio theory approach, in which bank choices of interbank debts and investments in non-financial firm assets are described as portfolio selection problems. Fundamentals that determine a bank network structure in equilibrium include bank capitals, the risks of bank debts, and banks’ risk aversions. In addition, I examine how financial stability is affected by the network structures of interbank debts and the common asset holdings of non-financial firms.

Financial intermediaries during the recent decades featured high leverage and complex interconnectedness through high frequency interbank activities and common asset holdings, which created large correlations between bank balance sheets. These features contribute to both the riskiness of individual financial institutions and the vulnerability of the entire banking system. According to Bernanke (2013a, 2013b), the financial crisis of 2007-08 was ignited by losses from the subprime mortgage market; weaknesses in the financial system transformed what might otherwise have been a modest recession into a much more severe crisis. Considerable authors conclude that high interconnectedness between banks provides a channel of risk propagation, which leads to the amplification of idiosyncratic shocks to banks (see in particular Allen and Gale (2000); Brusco and Castiglionesi (2007); Bernanke (2010); Acemoglu et al. (2013, 2015)).

This paper focuses on the network of interbank debts1 and banks’ common investments in non-financial firm assets, both of which provide potential channels of risk transmission.

To understand the contagion effect, it is essential to investigate how banks make decisions to enter into obligations to one another in the first place and how their decisions depend on fundamentals. It is also crucial for policy-making to examine bank decisions of interconnectedness, because financial fragility is intensified as bank decisions create externalities to counterparties that lead to an inefficient network structure. Such inefficiency arises when individual banks free-ride on the willingness of others to pay for financial stability and to provide liquidity (see Zigrand (2014)). Therefore, the price mechanism of interbank transactions will not fully reveal

\footnote{Gorton (2012) asserts that all financial crises involve runs on bank debts, which takes many different forms. In American history, for example, people run on private bank notes during crises; during the national banking era, there were runs on demand deposits; the financial crisis of 2007-08 is about repurchase agreements (repo), commercial paper, and prime broker balances.}
the underlying risks. Then government policy needs to be implemented to eliminate the systemic externalities and the inefficiency. Furthermore, analyzing banks’ behavior at the micro-level allows one to predict how banks will react to a change of government policy. When designing regulatory policies using an equilibrium micro theory, feedback effect has been taken into account, so there is no Lucas Critique.

A growing body of theoretical literature that models the formation of interbank lending is based on the work of Diamond and Dybvig (1983), and Allen and Gale (2000), in which banks strategically form links for risk sharing that is motivated by stochastic liquidity preference. Related papers include Castiglionesi and Navarro (2010), Babus (2013), and Wang (2015). Another body of literature, including Farboodi (2014), Bluhm et al. (2014) and Erol and Vohra (2014), models the motivation of banks to form links through their profit maximization process. Erol and Vohra (2014) focus on strategic default. In the literature theoretical models are limited by the flexibility with which they generate a variety of network structures. In Babus (2016), a bipartite network is formed between two regions of banks, and banks share the goal of minimizing the probability of system-wide default. In Farboodi (2014), banks are homogeneous, and network formation is driven by banks’ motivation to reach for investment opportunities. These models have a limited capacity to deal with large numbers of banks. Here, I develop a tractable model in which a general number of banks form two layers of networks – the network of interbank debts and common investments in non-financial firm assets. A variety of network structures under different values of fundamentals can emerge in my model, thereby providing flexibility for policy applications. Moreover, network formation models generally are difficult to be estimated empirically because of their high dimensions. In contrast, endogenous bank networks in my model depend on a few parameters, and this feature significantly decreases the estimation cost when the model is tested empirically.

My model adopts a mean-variance portfolio selection approach that is extended from Markowitz (1956). Banks that lend to counter-parties are described as making risky investments in the debtor banks’ debts. The risk lies in the possibility that the debtor banks might default, and is given as public information. To capture the externalities of inter-bank activities, I assume that bank default risks are believed to be independent of bank behaviors. In addition, banks are given opportunities to invest in firms outside the banking sector. Connections among banks are incentivized by their preferences to diversify investments in order to balance between expected
returns and risks of their investment portfolios. Network structure is then determined by the decisions of banks to issue debts, invest in other banks’ debts, and invest in non-financial firm assets. Also significant are the expected returns of bank debts, which are solved endogenously as market clearing factors. When the expected returns of non-financial firm assets are strictly positive, equilibrium of an interbank debt network always exist, and multiple equilibria are likely to emerge without normalization within expected returns of bank debts. In most of the cases, no link in the banking sector is one of the equilibria. In order to identify the most likely equilibrium bank network given correlations among bank debts, I adopt a one factor model to achieve an unique equilibrium. Equilibrium does not always exist under the factor model in which linear relationships on expected returns of bank debts narrow down the range of adjusting relative expected returns of bank debts.

I then conduct comparative statics of changing bank risk preferences in the mean-variance technology. Key results from computational examples with a small number of banks illustrate that given risks of bank debts, as banks become more risk tolerant (i.e., less risk averse), they tend to issue more debts and form more links within the banking sector. In other words, integration (bank’s dependence on counter-parties), diversification (number of counter-parties per bank), and the density of bank network increase. However, bank investment in the non-financial sector could follow a non-monotonic pattern. Robustness check indicates that the results hold for a variety of different network structures that emerge under distinct parameter values.

Another goal of this paper is to examine the impact of interconnectedness on financial stability. To achieve the goal, I compute probabilities of bank defaults and compare the contagion effects in different network structures. This paper add to the literature of analyzing risk propagation mechanism through financial networks that employ the fictitious default algorithm developed by Eisenberg and Noe (2001). Analysts extend the algorithm to meet different research needs. In this paper, also following Eisenberg and Noe (2001), I add to interbank debt network a second network layer of non-financial firm investments. Simulations are conducted in two examples: a circle network and a more connected network. I compare the systemic

\[\text{Network density is defined as the portion of the potential connections of all banks that are active connections (with strictly positive interbank transactions).}\]

\[\text{See, for example, Rogers and Veraart (2013), Elliott et al. (2014), Acemoglu et al. (2015), Duarte and Eisenbach (2015), and Glasserman and Young (2015).}\]

\[\text{For instance, Rogers and Veraart (2013) extend the algorithm by adding bankruptcy cost to explore how failing banks might be rescued by other banks in the financial sector; Elliott et al. (2014) add bank cross holdings and bankruptcy cost to study how propagation of bank failures depends on integration and diversification.}\]
stability of the two networks, which is measured as the probability that multiple banks will
default simultaneously. In each network structure, I study the impact of common asset holdings
and correlations among asset returns on systemic stability. Simulation results demonstrate that
higher network density, higher correlations among asset returns, and common asset holdings all
significantly contribute to a higher probability of systemic failure.

2 The Model

Consider a one-period model with a banking sector consisting of a set \( B = \{1, \ldots, N\} \) of banks,
and thus, a finite number of \( N \) nodes in the banking network. At the beginning time \( t = 0 \),
banks are endowed with exogenous bank capitals \( e = (e_1, \ldots, e_N)\) \( \gg 0 \) that can be heteroge-
neous. There is a set \( T = \{1, \ldots, K\} \) of firms outside the banking sector that all banks have
accessibilities to invest in. Firm asset returns are stochastic with given expected returns, var-
iances, and covariances among assets, which are common knowledge to banks. A bank finances
its investment through its own bank capital and by borrowing from other banks\(^5\). When a bank
borrows from other banks, it issues debt at time \( t = 0 \). Its creditor banks gain interest payments
from the debt at maturity date \( t = 1 \). Since a debtor bank might default on its payment at time
\( t = 1 \), a bank debt is risky. Assume that if a bank defaults, it will pay all the cash flow it has\(^6\)
proportionally to its creditor banks according to the sizes of the debts. Also assume limited
liability. Consequently, the realization of a bank debt payment as a percentage of its obligation
distributes from zero to one. Assume that the expected rate of returns on bank debts and firm
assets are strictly positive. Then banks will invest in bank debts or firm assets rather than hold
cash.

2.1 Banks’ Investment Portfolios

At time \( t = 0 \), each bank decides on its investments in \( N - 1 \)\(^7\) bank debts and \( K \) firms, and
each decides how much debt to issue to finance its investment, as if it forms a portfolio of the
\( N + K \) risky assets. Define \( q_b \geq 0 \) to be the total dollar amount of bank \( b \)'s borrowing or debt

---

5 Financing from outside the banking sector is excluded here in order to focus on activities within the banking system.

6 The cash flow a bank receives at \( t = 1 \) includes payments from its debtor banks and firms it invested at \( t = 0 \).

7 A bank cannot purchase its own debt.
issued, and $s_{ib} \geq 0$ to be the dollar amount that bank $b$ invested in asset $i$. Bank $b$’s balance sheet is shown in Table 1, for $b \in B$

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lending to Banks $\sum_{i \neq b} s_{ib}$</td>
<td>Borrowing from Banks $q_b$</td>
</tr>
<tr>
<td>Loans to Firms $\sum_{i \in T} s_{ib}$</td>
<td>Capital $e_b$</td>
</tr>
</tbody>
</table>

Table 1: Balance Sheet of Bank $b$

When a bank forms its investment portfolio, it assigns a non-positive weight of its own bank debt because the bank is taking a short position on its debt, and it assigns non-negative weights to other bank debts and firm assets. Portfolio weights are normalized by bank capital. For a bank $b \in B$, define its portfolio weights as

$$w_{ib} := \begin{cases} 
\frac{s_{ib}}{e_b} & \text{for } i \neq b, i \in B \cup T, \\
-\frac{q_b}{e_b} & \text{for } i = b, i \in B \cup T.
\end{cases}$$

Each element $w_{ib}$ in $W$ denotes bank $b$’s investment with respect to bank capital in asset $i$. Since banks do not hold cash, portfolio weights for each bank $w_b$ always add up to 1$^8$.

I denote the portfolio weights of all banks by $W = [w_1, \ldots, w_b, \ldots, w_N]$, where each column $w_b$, for $b \in B$, is a vector of each bank’s portfolio weights. Let $w_{Bb}$ be bank $b$’s portfolio weights on interbank debts, and let $w_{Tb}$ be portfolio weights on non-financial firm assets. Then $w_b = \begin{pmatrix} w_{Bb} \\ w_{Tb} \end{pmatrix}$ for bank $b$. $W$ can also be written as $W = \begin{pmatrix} W_B \\ W_T \end{pmatrix}$, where $W_B$ represents weights on bank debts and determines the structure of the banking network and $W_T$ represents weights on firm assets and captures the network of banks’ common investments outside the banking sector.

Assume that banks are in favor of expected returns but dislike variances$^9$. Let the rate of returns of bank debts and firm assets be jointly distributed with the mean $\mu(W) = \begin{pmatrix} \mu_B(W) \\ \mu_T \end{pmatrix}$

---

$^8$For a proof see Appendix A.

$^9$The mean-variance rule for preference can be used in three scenarios. In the first scenario, the investor’s utility function is quadratic. In the second, asset returns are normally distributed in the face of risk aversion. In the third, even in the absence of the first two assumptions, the mean-variance rule is a good approximation of expected utility when returns on the investment are not overly extreme. (See Levy (2012))
and an exogenous, non-singular variance-covariance of all assets, \( V \). \( \mu_B(W) \) represents the vector of expected returns of bank debts. For a given \( \mu_B \) and \( V \), each bank determines its optimal investment portfolio. To keep the demand and supply of each bank debt balanced to reach an equilibrium, contractual interest rate on the bank debt needs to be adjusted. Therefore, the expected returns of bank debts adjust as well, and endogenously depend on banks’ investment decisions \( W \) in equilibrium. Section 2.2 and section 2.3 provide a detailed explanation of how to solve for equilibrium. To keep non-financial firms simple, I assume, first, that the expected returns of firm assets are given by the fixed vector \( \mu_T \) and, second, that the supply of firm assets is perfectly elastic.

For simplicity, assume that at the beginning date \( t = 0 \), banks are not connected with non-financial firms, and that banks ignore the potential increase in bank-firm correlations that could take place when they invest in the firm assets. Also, assume that there is no common risk faced by the banking sector and the private sector. Then, the correlations between bank debt returns and firm asset returns are zeros\(^{10}\). Therefore, we have

\[
V = \begin{pmatrix}
V_B & 0 \\
0 & V_T
\end{pmatrix},
\]

where \( V_B \) is the variance-covariance of bank debt returns, and \( V_T \) is the variance-covariance of firm asset returns.

Moreover, assume that \( V \) is a prior estimated from historical information and that it is common knowledge across banks. The assumption that \( V \) is independent of portfolio choices \( W \) means that banks make decisions without considering how these decisions affect bank default risks and correlations, even though the links formed among banks could create vulnerability in the entire banking sector and potentially increase the actual bank default risks and correlations among institutions. Consequently, externalities will arise, and expected bank debt returns will not reveal their underlying risks, leading to an inefficient banking network.\(^{11}\)

\(^{10}\)One can assume non-zero correlations between banks and firms at \( t = 0 \) that are determined by historical lending relationships and common risks. In this situation, bank portfolio decisions will be different from the case of zero correlations. Nevertheless, the mechanism of the model to deliver an equilibrium bank network will remain the same. More specifically, the key factors that determine portfolio selections are the relative expected bank debt returns with respect to the expected firm asset returns. This result holds under the zero correlation assumption, as demonstrated in Proposition 2.2.

\(^{11}\)It is possible that banks aware that their behaviors will influence risks of counterparties that are directly connected to them (the first-order connections), and yet, banks are unlikely to be equipped with enough informa-
2.2 Banks’ Portfolio Selection Problem

Each bank chooses investment portfolio \( w_b \) to maximizes its equity value\(^{12}\) according to a mean-variance rule. Each bank’s efficient portfolio set can be represented by the solutions of the following portfolio selection problem: for any bank \( b \in B \),

\[
\min_{w_b} V^p_b(w_b) \equiv w_b^\top V w_b
\]
\[
s.t. \quad \mu(W)^\top w_b = \mu^p_b \quad (\lambda_b)
\]
\[
1^\top w_b = 1
\]
\[
w_{ib} \geq 0, i \neq b
\]
\[
w_{bb} \leq 0,
\]

where \( V^p_b(w_b) \) is variance of the portfolio of bank \( b \), and \( \mu^p_b \) is the expected return of the portfolio of bank \( b \). \( \lambda_b \) is the Lagrange multiplier of the first constraint, which is also the risk tolerance of bank \( b \) representing the trade-off between the expected return and the risk of bank \( b \) debt to keep its expected utility unchanged.

Problem 2 resembles the portfolio selection problem that Markowitz (1952) develops, and yet it is distinguished by two differences. First and most importantly, the expected returns of bank debts serve as market clearing factors of bank debts. They are endogenously determined by portfolio selections of all banks. That it, for given expected returns of bank debts \( \mu_B \), I trace out each bank’s efficient frontier by varying \( \mu^p_b \) from the expected return of the portfolio associated with minimum variance to the maximum feasible expected return. Each bank’s risk tolerance will determine the shape of its indifference curves, thus, its optimal portfolio. The demand and supply of each bank debt then are computed from the optimal portfolio choices made by all the banks. Next, the demand and supply curves of each bank debt are traced out by varying expected returns \( \mu_B(W) \) within a certain range. The \( \mu_B(W) \) that clear the markets

\(^{12}\)Therefore, when making a portfolio selection decision, the bank will take into consideration the probability that it will default on its debt, or, alternatively, the variance of its bank debt.
for all bank debts will be the equilibrium $\mu_B^*$, and we can solve for the equilibrium network structure $W^*$ associated with $\mu_B^*$.

The second difference consists of the last two constraints in problem 2 that prevent a bank from purchasing its own debt and short selling other assets. When I change subscript $b$ these constraints vary across banks, which indicates that banks have different efficient portfolio sets.

According to Markowitz (1956), the covariance matrix $V$ is non-singular if and only if the variance of each bank’s portfolio $V^p_b(w_b) = w_b^\top V w_b$ is positive definite, which in turn is true if and only if $V^p_b(w_b)$ is strictly convex over the set of all $w_b$. Let $X_b$ be any point in the space of choice variables $w_b$ for bank $b$, which satisfies the constraints in problem 2. The points that satisfy the constraints and have $V^p_b \leq V^p_b(X_b)$ form a compact, convex set. Moreover, $V^p_b(w_b)$ is continuous. Given the expected returns $\mu$ of all assets, the variance-covariance matrix $V$, and a bank’s risk tolerance $\lambda_b$, $\forall b \in B$, there always exist an unique minimum of $V^p_b(w_b)$ with portfolio choice $w_b$ for each bank’s optimal portfolio selection problem.

To trace out the efficient frontiers and solve for each bank’s optimal portfolio I adopt the critical line method developed by Markowitz (1956). Sharpe (1970, 1995) illustrates the computational algorithm used in a more general model set-up. Assume that banks’ expected utilities are

$$U_b(W) = \lambda_b \mu(W)^\top w_b - w_b^\top V w_b, \ b \in B,$$

Then, the banks’ problem 2 can be rewritten as: $\forall b \in B$,

$$\max_{w_b} U_b(W)$$

$$s.t. \quad 1^\top w_b = 1$$

$$lb_b \leq w_b \leq ub_b,$$

where $lb_b$ is the vector of the lower bounds on bank portfolio weights as percentage of bank capital, which prevents banks from short-selling. Similarly, $ub_b$ is the vector of the upper bounds of portfolio weights. Note that the boundary vectors differ across banks because the constraint $w_{bb} \leq 0$ differs across banks. An upper bound on bank debt represents the capital requirement assigned exogenously by a central bank, which is identical across banks. A bank’s efficient frontier can be traced out by varying its risk tolerance $\lambda_b$ from zero to infinity. In this quadratic problem with linear constraints, optimal portfolio weight of an asset that associate with the
bank’s risk tolerance $\lambda_b$ might fall into one of three statuses:

- **DOWN**: the lower bound of the portfolio weight is binding;
- **IN**: it falls within the lower bound and the upper bound (interior solution);
- **UP**: the upper bound of the portfolio weight is binding.

According to Markowitz (1956) and Sharpe (1970, 1995), the efficient frontier of each bank can be characterized by a finite number of corner portfolios. A corner portfolio is one that is optimal for a risk tolerance at which an asset weight changes status. The value of risk tolerance associated with the corner portfolio is called critical value. More specifically, we have the following lemma.

**Lemma 2.1. (C Fund Separation)**

1. In a mean-variance portfolio selection problem with inequality constraints, if there are $C$ different corner portfolios, then the minimum number of mutual funds required to serve all possible investors will equal $C$.
2. For an investor whose risk tolerance falls within two critical values, $\lambda_1$ and $\lambda_2$, his optimal portfolio can be constructed by taking a weighted average of the corner portfolios associated with $\lambda_1$ and $\lambda_2$, in which the weights are proportional to the difference between his risk tolerance and the two critical values.

**Proof.** See Markowitz (1956).

Lemma 2.1 suggests that given fixed expected returns of assets, efficient portfolio set for each bank consists of kinked lines against risk tolerance. When the expected bank debt returns adapt to reach a banking sector equilibrium, the shape of banks’ utilities $\{U_B(W)\}_{b \in B}$ and the corner portfolios change, as do the shape of the efficient portfolio frontiers. Appendix B provides the algorithm of the critical line method that I use to compute efficient frontiers; it incorporates a simple numerical example to illustrate Lemma 2.1.

Under the assumption of non-correlated bank debt returns and firm asset returns – as is indicated in the block-diagonal covariance matrix $V$ – a bank’s investments distributed within non-financial firm assets $w_{Tb}$ will be related with its portfolio weights distributed within inter-bank debts $w_{Bb}$. The intuition is that what determine banks’ portfolio selections are the relative expected returns of bank debts with respect to the expected firm asset returns, rather than the
absolute values of expected returns. Even though expected returns of firm assets are fixed, the relative expected returns of bank debts still vary as $\mu_B$ adapts.

**Proposition 2.2. (Interaction between $w_{Bb}$ and $w_{Tb}$)**

Assume $\sigma_{ib} = 0$ for any $i \in T$ and $b \in B$. In banks’ optimal portfolio selection problem 4 without short-selling, the portfolio weights distributed within the firm assets, $w_{Tb}$, is related to the portfolio weights distributed within the interbank debts, $w_{Bb}$. Furthermore, when all banks face an unique common mutual fund $t$, a risk-averse bank’s portfolio weight on a bank $j$ debt is positively related to $\frac{\mu_j}{\mu_t}$, for $j \in B$.

Proof. See Appendix C.

2.3 Banking Sector Equilibrium

The banking network in equilibrium is defined as follows:

**Definition 2.1. (Equilibrium Banking Network)** Given the expected returns on firm assets $\mu_T$, the variance-covariance matrix $V$ of $N$ bank debts and $K$ firm assets, bank capitals $\{e_b\}_{b \in B}$, and the risk tolerances of banks $\{\lambda_b\}_{b \in B}$, the network of inter-bank debts and banks’ investments in firms are determined by the portfolio matrix $W^*$ and the expected returns of bank debts $\mu_B(W^*)$, where $W^*$ and $\mu_B(W^*)$ solve the mean-variance optimization problem (2) for all banks with associated risk tolerances. Meanwhile, portfolio weights on interbank debts $W^*_B$ satisfy the market clearing conditions of bank debts:

$$\sum_{j \in B} w_{bj} e_j = 0, \quad \forall b \in B. \tag{5}$$

The market clearing conditions for bank debts are from $-w_{bb} \cdot e_b = \sum_{j \neq b} w_{bj} e_j$ for all $b \in B$; that is, the amount of a bank debt issued must equal the total amount demanded by all other banks. Appendix D provides the algorithm that I use to compute equilibrium.

Given the expected returns $\mu_T$ of firm assets, equilibrium in the banking sector does not always exist. From Proposition 2.2, we know that relative expected returns play a crucial role in determining a bank’s portfolio selection. If there is a firm asset dominates other firm assets in terms that the dominating firm asset has a strictly higher expected return but the variance of its
return is not greater than that of other firm assets, then equilibrium does not exist. Under the condition that there is no dominating firm assets, observations from computational examples show that equilibrium always exists, and usually multiple equilibria emerge. However, in many cases, banking sector reaches a continuum of equilibria when expected returns of all bank debts are identical, falling within a certain range. In such cases, banks have no incentive to borrow or lend, therefore, there is no link formed in the banking sector and banks are all isolated.

2.3.1 Equilibrium with One Non-Financial Firm

For the case in which there is one firm or one mutual fund faced by all banks, there will be no dominating firm asset. If expected returns of bank debts are not normalized within $\mu_B$, equilibrium always exists. In many cases, multiple equilibria exist, among which no borrowing and lending in the banking sector with identical expected returns of bank debts is always one equilibrium. For some parameter values, no bank connection is the only situation of equilibria, while for other cases, there could be an equilibrium with connections in the banking sector besides autarky. Based on the linear property of a bank’s efficient frontier shown in Lemma 2.1, a continuum of equilibria tends to emerge for the equilibria of no bank connection. The following example shows a situation of multiple equilibria of a banking sector with 2 banks.

Example 2.1. (Multiple Equilibria: One Non-Financial Firm)

Consider a case in which there are two banks and one firm asset. Assume that the boundary for bank debt is no greater than 100 times the bank capital. Let bank capitals be $e = [100, 100]^\top$. Let the expected return of the firm asset be $\mu_t = 0.13$. The variance-covariance matrix of bank debt and firm asset returns is given by

$$V = \begin{pmatrix}
0.009296 & 0.004536 & 0 \\
0.004536 & 0.019876 & 0 \\
0 & 0 & 0.0625
\end{pmatrix}.$$

Let the risk tolerance of bank 1 be 10, and that of bank 2 be 1. Adjust the expected returns of both bank debts from 0.00001 to 0.2. A continuum of equilibria emerges when the expected returns of both bank debts are identical and in the range $[11.75\%, 12.99\%]$. In this case, no active link is formed between the two banks. Another equilibrium exists when $\mu_1 = 10.93\%$.
and $\mu_2 = 10.67\%$. In this case, bank 1 borrows $57.01\%$ of its bank capital from bank 2. Panel (a) and (b) in Figure 1 show excess demand of bank 1’s debt and bank 2’s debt respectively. Because of the linear property of efficient frontiers, the supply and demand curves of bank debts consist of linear segments. The excess demand of a bank debt in turn consists of hyperplanes in the space spanned by the expected returns of bank 1’s debt and bank 2’s debt, and the amount of bank debt.

![Figure 1: Excess Demand of Bank Debts: Multiple Equilibria Exist](image)

(a) Excess Demand of Bank 1 Debt  
(b) Excess Demand of Bank 2 Debt

2.3.2 Identification of Equilibrium Network

Given the fixed belief of covariance matrix $V$, it is unlikely that the expected returns of bank debts can adjust in a large range without any normalization. In order to narrow down networks in equilibrium to the one that is likely to emerge given the correlations among bank debt returns in $V_B$, I adopt the one factor model to add constraints on how the expected returns of bank debts adjust. Assume that the returns of bank debts are correlated with a common factor $f$ with the given expected return $\mu_f$ and variance $\sigma_f^2$:

$$r_b = \alpha_b + \beta_b r_f + \epsilon_b, \quad \forall b \in B,$$

where $\epsilon_b$’s are error terms that are independent across banks with mean zero and are uncorrelated with $r_b$ for all $b \in B$. Then the expected returns and risks of bank debts are determined by:

$$\mu_b = \alpha_b + \beta_b \mu_f, \quad b \in B; \quad (6)$$
\[ \sigma_b^2 = \beta_b^2 \sigma_f^2 + \sigma_{eb}^2, \quad b \in B. \]  

Covariance of any two bank debts is given by

\[ \sigma_{ij} = \beta_i \beta_j \sigma_f^2, \quad i \neq j, \quad i, j \in B. \]  

The expected return of each bank debt has a linear relationship with the factor, and this relationship will hold when we vary \( \mu_B(W) \) to reach banking sector equilibrium. Therefore, to reach an equilibrium we only need to vary \( \mu_f \). The risks of the returns of bank debts are determined by slope coefficients, the variance of the factor, and idiosyncratic risks. Thus, the variance-covariance matrix \( V_B \) also can be calculated using equations (7) and (8). Assume that bank debts bear distinct expected returns and risks. Then I expect each bank to have a diversified portfolio under appropriate parameter values\(^\text{13}\).

The linear constraints (6) on the expected returns of bank debts are not too strong. First, banks, regardless of their sizes, businesses, geographic locations, etc., all face the similar macroeconomic environment and regulatory policies. Therefore, it is nature to assume that there is a common factor that drives the expected return of bank debts. Second, restrictions on the first and second moments of bank debt returns will not limit the types of network structures that can emerge in equilibrium.

Simulations show that equilibrium does not always exist when the factor model is employed to identify equilibrium bank network. Note that based on Lemma 2.1, in the quadratic problem (2) with constraints, the demand curve and the supply curve of each bank debt are kinked. The non-existence of equilibrium occurs in two possible cases. In the first case there is no equilibrium in the market of at least one bank debt. In other words, when adjusting the expected return of the factor in the range \( (0, 1) \), there is no intersection of demand and supply curves. This could occur when the risk tolerance of a bank is so low that it is reluctant to issue debt; at the same time, the risk tolerances of other banks are sufficiently large that the demand of its bank debt is strictly positive. In the second case, following the linear relationships in the factor model, there is no expected returns of bank debts that can clear markets of all bank debts simultaneously. Hence, the existence of equilibrium is restricted by adjusting the expected returns of bank debts

\(^{13}\)Parameters in the model include endowments \( e \), the variance-covariance matrix of all assets \( V \), and the expected returns of firm assets \( \mu_T \).
with the linear relationships from the factor model.

The benefit of fixing the linear relationships of expected bank debt returns when solving for equilibrium is as follows: given that equilibrium bank network exists, the equilibrium is unique. For the following comparative statics experiments, I adopt the factor model to achieve unique equilibrium bank networks.

## 3 Comparative Statics of Network Formation

For the sake of simplicity, I conduct comparative statics of adjusting banks’ risk tolerances in the case of one non-financial firm. Comparative statics in the case of five banks and one firm are similar to the case of four banks. To reduce computation costs, I examine examples of four banks and one firm. In the following examples, I increase the risk tolerances of all banks. Meanwhile, given parameter values, I pick combinations of banks’ risk tolerances to satisfy the requirement that there is an equilibrium of bank networks.

### Example 3.1. (Incomplete Network)

Consider a banking sector consists of four banks which are of the same bank capital. Each bank has ten million dollars of bank capital. Therefore, \( e = [10, 10, 10, 10]^{\top} \). Set the boundary for bank debt so that it is not greater than 100 times its bank capital; thus, the upper bound for leverage is unbinding. Let the expected return of the factor \( \mu_f \) vary in a range \([0.01\%, 50\%]\) and the standard deviation of the factor be \( \sigma_f = 18\% \).\(^{14}\) In the factor model, the expected returns of bank debts are assumed to follow linear relationships:

\[
\begin{align*}
\mu_1 &= 0.0011 + 0.6\mu_f, \\
\mu_2 &= 0.0011 + 0.6\mu_f, \\
\mu_3 &= 0.0013 + 0.601\mu_f, \\
\mu_4 &= 0.0012 + 0.601\mu_f.
\end{align*}
\]

Variances of the residuals are \( \sigma_e^2 = [0.001, 0.005, 0.0051, 0.008]^{\top} \). Firm asset has a expected

\(^{14}\)How to choose the common factor can be analyzed in future empirical work. Potential candidates include typical macroeconomic indicators and financial market indicators. For anyone interested in studying banks’ historical performances, note that stock prices of banks may not be a good choice because from 1873 through the early 1960s bank stocks did not actively trade even though they remained public (see Gorton(2014)).
return of 2.5% and a standard deviation of 7%. Then one can compute $V$ according to equations (9) to (12). Increase the risk tolerance of bank 1 from 1 to 4.5; the risk tolerance of bank 2 from 1.5 to 5; the risk tolerance of bank 3 from 2.5 to 500.5; and the risk tolerance of bank 4 from 2.11 to 45. Risk tolerances increase in a non-linear pattern to assure the existence of an equilibrium network.

In equilibrium, banks form an incomplete network\textsuperscript{15}, as shown in Figure 2a. Each directed edge represents cash flow or a long position on bank debt at time $t = 0$. As bank risk tolerances grow, in equilibrium, expected bank debt returns increase, as displayed in Figure 2b. When the four banks’ risk tolerances increase to $[1.9423, 2.4423, 16.5, 11.08]$ respectively, bank 1 and 2 are pure lenders. Bank 3 and 4 are pure borrowers: they take higher leverage and invest more in the firm asset as banks become more risk tolerant.

As risk tolerances rise above $[2.01, 2.51, 17.5, 11.6]$ respectively for the four banks, bank 4 starts to lend to bank 3 and, hence, there is one more active link formed in the banking network. Bank 3’s debt increases to about 60% of its bank capital as its risk tolerance goes to 500, which approaches risk neutrality.\textsuperscript{16} Banks 1 and 2 lend more to banks 3 and 4. Consequently, integration increases as debtor banks depend more heavily on financing within the bank sector; diversification increases particularly for bank 3 whose debt is held by more creditor banks; and the density of the network increases as numbers of counter-parties increase for banks 3 and 4.

Unlike inter-bank lending, banks’ investments in the non-financial firm follow a non-monotone pattern, particularly for bank 4, as shown in Figure 2f. Bank 3 invests more in the firm asset, while banks 1 and 2 reallocate their portfolio weights from the firm asset to bank debts. Besides, the total amount invested in the firm by the entire banking sector also follows a non-monotonic pattern as bank risk tolerances rise. This result arises in different network structures as shown in Example 3.2 and Appendix E. The non-monotonic relationship between bank risk tolerance and investment in firm asset may suggest that bank’s risk preference is not a key factor to explain the pro-cyclical investment or credit crunch during financial crises. Other factors that have been kept constant in the comparative statics may play a much more crucial role, such as bank capital\textsuperscript{17}, bank debt risk, etc.

\textsuperscript{15}A complete network is one in which all pairs of nodes are connected, that is, a graph that is fully connected. An incomplete network is one in which there is missing edge or link between pair of nodes.

\textsuperscript{16}As other banks’ risk tolerances do not increase such rapidly in order to keep the existence of equilibrium, equilibrium interbank lending does not grow to a great extent.

\textsuperscript{17}There is strong empirical evidence demonstrates that bank equity and Tier 1 capital, as well as asset value
Figure 2: Comparative Statics of Four Banks: Incomplete Network

Note that in this example, bank 1 debt dominates bank 2 debt: both have the same expected influence credit provided to firms. See, for example, Popov and Udell (2012) for performance of the European banking industry during the 2007-08 global financial crisis, and Bernanke and Lown (1991) for the credit crunch during the recession of 1990 in the United States.
returns but bank 1 debt is less risky. In the real world, banks have access to different investment opportunities, so it is possible that one bank debt could dominate another bank debts.

**Example 3.2. (Star Network)**

Again, consider a banking sector consists of four banks. Assume Bank 1 has the largest amount of bank capital. \( e = [50, 10, 10, 10]^T \). Set the boundary for bank debt so that it is no greater than 100 times its bank capital. Let the expected return of the factor \( \mu_f \) varies in a range \([0.01\%, 50\%]\) and standard deviation of the factor be \( \sigma_f = 18\% \). In the factor model, the expected returns of bank debts are assumed to follow linear relationships:

\[
\begin{align*}
\mu_1 &= 0.005 + 1.7\mu_f, \\
\mu_2 &= 0.001 + 0.711\mu_f, \\
\mu_3 &= 0.001 + 0.71\mu_f, \\
\mu_4 &= 0.001 + 0.7\mu_f.
\end{align*}
\]

The variances of the residuals are \( \sigma^2_\epsilon = [0.03, 0.06, 0.055, 0.05]^T \). The firm asset has a expected return of 2.5% and a standard deviation of 7%. Increase the risk tolerances of banks 2, 3, and 4 from 0.01 to 0.4 linearly. To ensure the existence of equilibrium, increase the risk tolerance of bank 1 from 1 to 14 in a non-linear pattern.

In equilibrium, banks form a star network with bank 1 at its center (see Figure 3a). Bank 1 increases its debt from 1.135% to 2.179% of its bank capital. When bank 1 is endowed with 50 million dollars of bank capital, it increases its debt by 522,000 dollars. Banks 2, 3, and 4 increase their lending to bank 1 from about 1.89% to about 3.63% of each bank’s capital. In this example, bank 1 debt has the highest standard deviation – 35.16% – of all bank debts. However, its expected return is great deal higher than that of other bank debts, which causes all other banks to invest in bank 1’s debt. The star network structure further increases the fragility of the banking sector because the failure of bank 1 could cause the entire banking sector to collapse.

Unlike example 3.1, the network structure in the banking sector does not change. Only the density of the interbank lending network increase.

If one changes the bank capital of bank 1 so that it becomes the smallest, bank 1 can still remain at the center of a star network if appropriate parameter values are applied. For a bank
to be “too-connected-to-fail”, it does not need to have a large amount of bank capital compared to other banks.

Figure 3: Comparative Statics of Four Banks: Star Network
4 Financial Contagion

Equilibrium portfolio selections $W^* = \begin{pmatrix} W^*_B \\ W^*_T \end{pmatrix}$ determine the network of inter-bank debt and investments in firm assets. Let us now turn to the question that how financial stability depends on network structures. In this section, I compute equilibrium bank payments at time $t = 1$ when asset returns are realized, and compare probabilities of bank defaults in different network structures.

4.1 Bank Payment and Clearing Equilibrium

Given the realized returns of firms assets, one can identify banks that are directly affected by low cash flows received from firms they invested in. Consequently, one can identify banks that need to default as well as their default amount. Furthermore, banks that purchased the debts of defaulting banks also might need to default, and so on. In equilibrium, chains of defaults and equilibrium payments ensure that no further defaults will happen. The “fictitious default algorithm” developed by Eisenberg and Noe (2001) can be employed to track this domino effect and compute the equilibrium realized bank payments.

To compute the realized payments, we must first define a few terms. Let $I = (I_1, \ldots, I_N)^\top$ be the vector of total dollar amounts invested in firm assets by each bank. $I$ can be computed from portfolio weights $W^*$ and bank capitals $e$. Let $L = (L_1, \ldots, L_N)^\top$ be the vector of the total nominal contractual obligations of each bank to repay all of its creditors. $L$ can be computed from portfolio weights $W^*$, bank capitals $e$, expected returns $\mu(W^*)$, and the variance-covariance matrix $V$. For computational convenience, separate the inter-bank debt network from the firm investment network in matrix $W^*$. Re-normalizing weights in $W^*_B$, we get a matrix $M$ in which each non-zero row adds up to 1; this represents the inter-bank debt network. Similarly, re-normalize weights in $W^*_T$ to get matrix $A$ in which each non-zero row adds up to 1; $A$ reveals banks’ investment portfolios on firm assets.

Assume that banks’ realized payments towards their debt obligations are $p = (p_1, \ldots, p_N)^\top$ at time $t = 1$. For banks’ payment rule under default, I follow Eisenberg and Noe (2001):

1. Proportionality: If bank $b$ defaults, all its creditor banks are paid in proportion to the size of their nominal loans to bank $b$ with the same priority. For example, for one of its creditors, bank $j$, it will receive a proportion $m_{bj}$ of bank $b$’s payment, wherein $m_{bj}$ is the
element at row $b$ and column $j$ in matrix $M$.

2. Limited Liability: Each bank cannot pay more than its total cash inflow. $\forall b \in B,$

$$p_b \leq \sum_k a_{bk} r_{Tk} I_b + \sum_j m_{jb} p_j + e_b,$$

where $r_T$ is the vector of the realized rate of returns from firm assets. The first term on the right-hand side of the inequality consists of payments of firms assets. The second term is realized payments received from bank debts that bank $b$ invested at $t = 0$.

3. Absolute Priority: Either obligations are paid in full or all the operating income of a bank is paid to its creditors. $\forall b \in B,$

$$p_b = \begin{cases} L_b & \text{if bank } b \text{ does not default,} \\ \sum_k a_{bk} r_{Tk} I_b + \sum_j m_{jb} p_j + e_b & \text{if bank } b \text{ defaults.} \end{cases}$$

Assume that the payment vector in equilibrium is $p^*(r_T, A, I, M, L)$, which depends on realized returns of firm assets $r_T$ and the networks of inter-bank debt and investments in firm assets. Under default, and according to the three rules just specified, the clearing payment vector satisfies:

$$p_b = \min \{ \sum_k a_{bk} r_{Tk} I_b + \sum_j m_{jb} p_j + e_b, L_b \}, \quad \forall b \in B.$$

In matrix form:

$$p^* = \Phi(A r_T \circ I + M^T p^* + e, L) := \min \{ A r_T \circ I + M^T p^* + e, L \}, \quad (17)$$

where “$\circ$” is the Hadamard product\(^{18}\). Therefore, the clearing equilibrium payment vector $p^*$ is the fixed point of the mapping $\Phi(\cdot)$. A clearing equilibrium payment vector $p^*$ exists and it is unique for all network structures $A$, $I$, $M$, and $L$, provided that the asset rates of return are strictly positive with probability one. Appendix F provides the algorithm that is used to compute the clearing equilibrium payment.

---

\(^{18}\)The Hadamard product multiplies elements $ij$ in two matrices that are located in the same row $i$ and column $j$. In other words, for matrices $X$ and $Y$ of the same dimensions, $(X \circ Y)_{ij} = (X)_{ij}(Y)_{ij}$. 

20
4.2 Contagion in Bank Networks

Use the fictitious default algorithm, I compute bank default probabilities in two distinct hypothetical bank networks: a circle network and a more connected network. Both networks are incomplete and consist of five banks as displayed in Figure 4. The number besides each bank represents its bank capital, and the number on each directed edge represents a payment obligation or amount due at time $t = 1$. For instance, consider bank 2 in the circle network. It has bank capital of 31; it owes 54 to bank 1; it is owed 45 by bank 3; and it is owed 40 by firms outside the banking sector.

![Circle Network](image1)

(a) Circle Network

![More Connected Network](image2)

(b) More Connected Network

Figure 4: Payment Obligations in Two Simple Networks: Circle Network vs. More Connected Network

Across the two networks, I keep each bank’s investment in non-financial firms identical, that is, firm investment vector $I$ is the same across the two networks. I also keep total payment obligation of each bank, $L_b$ identical across the two networks. For example, bank 4 in both networks has a total obligation of 36 and is owed 30 by non-financial firms. In the circle network, bank 3 is its only creditor, while in the more connected network, bank 4 borrows from banks 1 and 3.

In the following examples, I compare bank default probabilities in the two networks and in two cases: diversified investments in non-financial firms, and common asset holdings in non-financial firms.

Example 4.1. (Diversified Non-Financial Firm Investment: Uncorrelated vs. Correlated Firm Assets)

Assume that in both networks all banks invest in different non-financial firm assets or projects,
so there are no common firm asset holdings. I call these firms “local firms.”

I conduct two experiments. The first independently draw 100,000 returns of non-financial firm assets from the normal distribution \( N(1.1, 0.3) \). The second draws 100,000 returns of firm assets from the same distribution, but with a correlation coefficient of 0.8. Table 2 shows the probabilities of bank payment defaults in the two networks. Bank 1 is excluded from the table because it is a pure lender in both networks. When there is positive correlation among firm assets, the probabilities of bank default increase in both networks. The circle network has lower probabilities of bank default compared to the more connected network, particularly in the case of banks 2 and 3. Bank 5 has the highest default probability in both networks because it is a pure borrower and takes the highest leverage. More connections in the banking sector widen the spread of risk, causing a contagion effect.

Table 2: Probability of Default: Uncorrelated vs. Correlated Firm Assets

<table>
<thead>
<tr>
<th>Default Bank</th>
<th>Correlation = 0</th>
<th>Correlation = 0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circle Network</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bank 2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Bank 3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Bank 4</td>
<td>0.002%</td>
<td>0.78%</td>
</tr>
<tr>
<td>Bank 5</td>
<td>25.43%</td>
<td>35.55%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>More Connected Network</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bank 2</td>
</tr>
<tr>
<td>Bank 3</td>
</tr>
<tr>
<td>Bank 4</td>
</tr>
<tr>
<td>Bank 5</td>
</tr>
</tbody>
</table>

However, a central banker will concern systemic failure instead of individual bank failure. In order to capture the probability of systemic failure, I compute the probability of multiple banks default simultaneously, as shown in Table 3. Banks 2 and 3 never default in the circle network, and so I report the probability of banks 4 and 5 default at the same time. For the more-connected network, I report the probability of banks 2, 3, 5 default simultaneously and banks 2, 3, 4, 5 default simultaneously. Default probabilities increase significantly when asset returns have correlations equal to 0.8.
### Table 3: Contagion Effect: Probability of Simultaneous Bank Defaults

<table>
<thead>
<tr>
<th></th>
<th>Uncorrelated</th>
<th>Correlation = 0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Circle Network</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bank 4,5</td>
<td>0.001%</td>
<td>0.782%</td>
</tr>
<tr>
<td><strong>More Connected Network</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bank 2,3,5</td>
<td>0.018%</td>
<td>0.879%</td>
</tr>
<tr>
<td><strong>More Connected Network</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bank 2,3,4,5</td>
<td>0</td>
<td>0.045%</td>
</tr>
</tbody>
</table>

Besides the contagion effect, another essential component of systemic risk is the amount of losses conditional on bank defaults. Figures 5 and 6 display the conditional distributions of the default amount as percentage of total obligation given bank defaults. In both networks, in the case where asset return correlations equal to 0.8, distributions generally have fatter tails than those in the case where asset returns are uncorrelated. It means that the probability of large amount of losses increases when asset returns are more correlated.

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19 See Glasserman and Young (2015).

---

**Figure 5: Default Distribution: Diversified Uncorrelated Firm Investments**
Example 4.2. (Common Asset Holding vs. Diversified Investments in Non-Financial Firms)

Consider the same two interbank lending networks from Example 4.1. I draw 100,000 returns of non-financial firm assets from the normal distribution \( N(1,1.1) \) with a correlation of 0.8. In this case, firm assets are less risky than those in Example 4.1. I conduct two experiments. The first one is similar to Example 4.1 in that banks diversify their investments in five local projects. In the second experiment, besides the 5 local projects, I add a 6th project that all banks can invest in. Assume that each bank allocates 30% of its private sector investment for the associated local project and 70% to the 6th project. Then the five banks have common asset...
holdings in the 6th firm asset.

Table 4 displays the probabilities of bank default. Similar to Example 4.1, banks 2 and 3 have a zero probability of defaulting. Bank 4, too, has a very low default probability. The default probabilities of individual banks are higher in the more connected network than those in the circle network for banks 2 and 3; there is no significant difference for other banks. In the more connected network, Banks 2, 3, and 4 have higher default probabilities when they have common holdings of the sixth bank’s assets than they do in the diversified investment case.

Table 4: Probability of Default: Diversified Investments vs. Common Asset Holding (Low Risk Firm Assets)

<table>
<thead>
<tr>
<th>Bank</th>
<th>Diversified Investments</th>
<th>Common Asset Holding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circle</td>
<td>Bank 2</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Bank 3</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Bank 4</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Bank 5</td>
<td>20.14%</td>
</tr>
<tr>
<td>More Connected Network</td>
<td>Bank 2</td>
<td>0.46%</td>
</tr>
<tr>
<td></td>
<td>Bank 3</td>
<td>0.23%</td>
</tr>
<tr>
<td></td>
<td>Bank 4</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Bank 5</td>
<td>20.14%</td>
</tr>
</tbody>
</table>

The probabilities of simultaneous defaults are displayed in Table 5. Although common asset holding does not have an obvious impact on the default probability of individual banks, it does significantly affect simultaneous default in both networks.

Table 5: Contagion Effect: Probability of Simultaneous Bank Defaults (Low Risk Firm Assets)

<table>
<thead>
<tr>
<th>Bank</th>
<th>Diversified Investments</th>
<th>Common Asset Holding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circle</td>
<td>Bank 4,5</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Default</td>
<td>0.006%</td>
</tr>
<tr>
<td>More Connected Network</td>
<td>Bank 2,3,5</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Default</td>
<td>0.222%</td>
</tr>
<tr>
<td></td>
<td>Bank 2,3,4,5</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Default</td>
<td>0.004%</td>
</tr>
</tbody>
</table>
In this experiment, I increase the standard deviation of firm assets to 40% while, as done above, I keep other parameters unchanged. Table 6 shows the default probabilities of the two networks in two cases. Again, banks in the more connected network have higher default probabilities. When banks have common investment, their default probabilities are slightly higher than in the diversified investment case. In the more connected network the impact of common asset holdings is obvious only in the case of bank 2.

Table 6: Probability of Default: Diversified Investments vs. Common Asset Holding (High Risk Firm Assets)

<table>
<thead>
<tr>
<th>Default Bank</th>
<th>Diversified Investments</th>
<th>Common Asset Holding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circle</td>
<td>Bank 2</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Bank 3</td>
<td>0</td>
</tr>
<tr>
<td>Network</td>
<td>Bank 4</td>
<td>0.84%</td>
</tr>
<tr>
<td></td>
<td>Bank 5</td>
<td>32.84%</td>
</tr>
<tr>
<td>More</td>
<td>Bank 2</td>
<td>8.96%</td>
</tr>
<tr>
<td>Connected</td>
<td>Bank 3</td>
<td>5.54%</td>
</tr>
<tr>
<td>Network</td>
<td>Bank 4</td>
<td>0.84%</td>
</tr>
<tr>
<td></td>
<td>Bank 5</td>
<td>32.84%</td>
</tr>
</tbody>
</table>

Table 7 displays the probabilities that simultaneous defaults will occur. As in the case of low risk firm assets, the probabilities of simultaneous defaults increase significantly when banks hold common assets.

Table 7: Contagion Effect: Probability of Simultaneous Bank Defaults (High Risk Firm Assets)

<table>
<thead>
<tr>
<th>Circle Network</th>
<th>Diversified Investments</th>
<th>Common Asset Holding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bank 4,5</td>
<td>0.084%</td>
<td>0.883%</td>
</tr>
</tbody>
</table>

| More Connected Network | Diversified Investments | Common Asset Holding |
| Bank 2,3,5 Default    | 0.305%                  | 4.638%               |
| Bank 2,3,4,5 Default  | 0.005%                  | 0.575%               |
Judging from these examples, when network structure is incomplete, asset return correlations and more connections in general contribute to instability and financial contagion. The common asset holdings of banks decrease stability to a small extent in the case of individual banks, but they increase significantly the probability of systemic failure.

5 Prospective Policy Implications

The results from comparative statics are consistent with empirical observations. A key feature of the financial cycle is that leverage taken by financial institutions and their inter-connectedness are pro-cyclical. During financial booms financial institutions are highly leveraged and connected, which increases their vulnerability. From a policy maker’s viewpoint, forming a highly connected network creates inefficiency in the financial sector. However, when banks decide to form such network, they do not realize that their behavior contributes to systemic instability or that forming a highly connected network might not be advantageous to individual banks. When subsequent losses and defaults occur during financial downturns, financial institutions start to behave cautiously. They stop rolling over debts within the financial sector, they hold cash, they become much less connected, and they provide insufficient liquidity in the financial market. Baklanova, Copeland, and McCaughrin (2015), for example, estimate that during the 2007-08 financial crisis, the daily average volume of securities lending peaked at about $2.5 trillion in April 2008 and then after January 2009 fell to about $1 trillion. Daily total repo activity was around $5 trillion, and this decreased substantially after the financial crisis.

My model can explain this phenomenon. During financial booms, banks are optimistic and they have a high risk tolerance. When their risk tolerances are high, they tend to issue more bank debts and depend on more diversified counter-parties, leading to a network with greater density. This network structure increases the probability that multiple banks will default simultaneously, and this, in turn, increases the chance that a financial crisis will occur. On the other hand, during financial downturns, banks become more risk averse, and, thus, they form less connected networks.

Furthermore, the model developed in the paper can be extended and applied to predict the effect of government policies on network structure. For example, in order to investigate the impact of a policy that changes banks’ assessment of the risks and correlations of bank debts, one can modify the assumptions about the covariance matrix \( V \) or coefficients in the factor model to
be endogenously dependent on government policies. Similarly, to examine the effect of capital requirements, one can vary the upper bound $ub_b$ of each bank’s portfolio weights. When the upper bound of interbank lending is binding for some banks, a network will emerge that consists of banks that borrow at the boundary. To hint at the potential implications of the model, I provide a simple example below of binding capital requirement.

**Example 5.1. (Binding Capital Requirement)**

Consider a simple example of two banks and one non-financial firm. Assume that the two banks have heterogeneous capitals $e = [100, 200]^T$. Let the upper bound to issue bank debt be 100% of bank capital. Let the expected return of the factor $\mu_f$ vary in a range $[0.01\%, 50\%]$ and the standard deviation of the factor be $\sigma_f = 18\%$. In the factor model, the expected returns of bank debts are assumed to follow linear relationships:

$$\mu_1 = 0.011 + 0.8\mu_f,$$

$$\mu_2 = 0.01 + 0.7\mu_f,$$

Variances of the residuals are assumed to be $\sigma_\varepsilon^2 = [0.02, 0.004]^T$. The firm asset has an expected return of 13% and a standard deviation of 25%. Assume that risk tolerance of bank 2 is 0.8. Increase the risk tolerance of bank 1 linearly from 2 to 70.

As displayed in Figure 7a, equilibrium expected bank debt returns increase as bank 1 becomes more risk tolerant. Figures 7b and 7c demonstrate that bank 1 borrows from bank 2, and both banks invest in the non-financial firm asset. Bank 1 borrows more and invests more in the non-financial firm asset when the capital requirement is not binding. However, as bank 1’s risk tolerance rises above 24.38, the capital requirement turns binding. As a result, the expected bank debt returns and portfolio weights for both banks remain unchanged beyond this point. Under the binding capital requirement, bank 1 borrows as much as it can - 100% of its bank capital - to finance its investment in the non-financial firm asset. Bank 2 invest a half of its capital in the non-financial firm asset and the other half to purchase bank 1 debt.
6 Further Discussions

It is not the aim of this paper to discuss how bank interconnections are affected by central bank intervention. Banks behave differently if they expect the central bank to serve as a lender of last resort. For three reasons the central bank is not key to understanding how financial crises happen. First, financial crises took place long before the existence of a central bank. During the National Banking Era in the United States, for instance, banks relied on each other for liquidity sharing. Second, with or without the existence of a central bank, financial crises in history have shared a common feature: sudden large demands for cash in exchange for short-term bank debt.

20For example, moral hazard issue arises with the existence of FDIC and banks tend to make riskier investments if they believe that central bank will bail them out during panics.
The existence of a central bank makes financial crises more complicated but it does not change the structure of crises. Third, it is imperative that before intervening in the banking sector or imposing regulation policies, the central bank must understand how banks act and what leads to crises. To study the effect of government interventions on bank networks, a central bank can be added to my model.

For researchers who are interested in studying banks’ fly-to-quality behavior during panics, a risk-free government bond need to be included. For those who want to analyze the contribution of fire-sale process to losses through bank networks, a dynamic model, or at least a two-period model will be needed to describe change of asset prices during fire-sales. In addition, my model can be extended to handle multiple layers of networks and thus take into consideration other investment types.

7 Conclusion

In this paper, I employ portfolio theory to propose a model of bank network formation in the case that banks do not consider the potential systemic externality and inefficiency generated from their interconnectedness. Interbank lending network and banks’ investments in the private sector are determined by bank capitals, bank debt risks, and banks’ risk tolerances. In a banking sector with a small number of banks, an increase in risk tolerances leads to higher integration, more diversification, and greater density of the banking network. However, investments in non-financial firm assets do not monotonically depend on risk aversions.

Simulation examples of the circle and the more connected network demonstrate that when the network is incomplete, while keeping payment obligations unchanged for individual banks, more connections in a bank network significantly increases contagion effect and banking system vulnerability, with or without common asset holdings. In addition, the common asset holdings of non-financial firm assets, as well as higher correlations among asset returns, contribute significantly to systemic fragility in both networks.
Appendix A  
Bank Portfolio Weights Add up to 1

Since the expected rate of returns on bank debts and firm assets are strictly positive, it is optimal for a bank to invest all its cash instead of holding it. At time $t = 0$ for any bank $b \in B$, its investments in $N - 1$ bank debts and $K$ firms are financed by its endowment $e_b$ and by borrowing from other banks. Therefore,

$$e_b + q_b = s_{1b} + \cdots + s_{b-1,b} + s_{b+1,b} + \cdots + s_{N+K,b} = \sum_{i \neq b \atop i \in B \cup T} s_{ib},$$

where $q_b$ is the dollar amount of bank $b$’s borrowing, or the debt issued, and $s_{ib}$ is dollar amount invested in asset $i$. Note that I exclude $s_{bb}$ on the right-hand-side of the equation because bank $b$ cannot purchase its own debt. Instead, the bank issues debt to raise fund $q_b$.

Rearranging terms, we have:

$$\sum_{i \neq b \atop i \in B \cup T} s_{ib} - q_b = e_b. \quad (A-1)$$

On the left-hand side of equation (A-1), we have all the assets that form a portfolio of bank $b$. These consist of investments in $N - 1$ bank debts and $K$ firm assets represented as the first term, as well as debt, $-q_b$ as a short position, that the bank itself issues. Its endowment is on the right-hand side. Assume $e_b > 0$, $\forall b \in B$. Normalizing by bank $b$’s endowment the dollar amount of the portfolio, we have:

$$\sum_{i \neq b \atop i \in B \cup T} \frac{s_{ib}}{e_b} - \frac{q_b}{e_b} = 1.$$

Portfolio weights are defined as

$$w_{ib} := \begin{cases} 
\frac{s_{ib}}{e_b} & \text{for } i \neq b, i \in B \cup T, \\
-\frac{q_b}{e_b} & \text{for } i = b, i \in B \cup T. 
\end{cases}$$

Then we have:

$$\sum_{i=1}^{N+K} w_{ib} = 1, \quad (A-2)$$

where $w_{ib} \geq 0$, for $i \neq b$, and $w_{bb} \leq 0$.

Equation (A-2) shows that the elements in vector $w_b$ that are the portfolio weights of bank $b$
Appendix B The Critical Line Method

This algorithm follows Markowitz (1956) and Sharpe (1995).

Consider the efficient portfolio frontier problem (4). For \( b \in B \), the Lagrange function is

\[
\mathcal{L}_b = \lambda_b \mu^\top w_b - w_b^\top V w_b - \gamma_b 1^\top w_b + \gamma_b,
\]

where \( \gamma_b \) is a multiplier of the constraint that requires that portfolio weights added up to 1 for each bank. Note that we need to solve for the efficient frontier with every value of expected asset returns \( \mu \). Consequently, the critical line method is applied for every fixed \( \mu \).

\( w_b \) is an optimal solution of the problem for bank \( b \) if and only if there exists \( \gamma_b \), which satisfies the following Kuhn-Tucker conditions:

\[
\frac{d\mathcal{L}_b}{dw_b} = \lambda_b \mu - 2V w_b - \gamma_b 1 = 0, \quad (A-3)
\]

\[
1^\top w_b = 1, \quad (A-4)
\]

\[
lb_b \leq w_b \leq ub_b. \quad (A-5)
\]

Putting conditions (A-3) and (A-4) in matrix form, we have:

\[
Dy = k + \lambda_b f, \quad (A-6)
\]

where \( D = \begin{pmatrix} 2V & 1 \\ 1^\top & 0 \end{pmatrix} \), \( y = \begin{pmatrix} w_b \\ \gamma_b \end{pmatrix} \), \( k = (0, \ldots, 0, 1)^\top \), and \( f = \begin{pmatrix} \mu \\ 0 \end{pmatrix} \).

Solving for \( y \):

\[
y = D^{-1}k + \lambda_b D^{-1} f. \quad (A-7)
\]

The computational algorithm is shown in the following steps:

1. Initially assume \( \lambda_b = \infty \) (as if the investor does not care about risks). Then problem (4)
becomes:

\[
\max_{\mathbf{w}_b} \mu^\top \mathbf{w}_b \\
\text{s.t. } \mathbf{1}^\top \mathbf{w}_b = 1 \\
\mathbf{l}_b \leq \mathbf{w}_b \leq \mathbf{u}_b.
\]

Solve for optimal solution \( \mathbf{w}_{b_{\inf}} \) of this linear programming problem.

2. Determine the status of weights in \( \mathbf{w}_{b_{\inf}} \) for each asset \( i = 1, 2, \ldots, N + 1 \), and divide the asset index into three sets:

\[
\begin{align*}
DOWN &= \{i | \mathbf{l}_b(i) = \mathbf{w}_{b_{\inf}}(i)\}; \\
IN &= \{i | \mathbf{l}_b(i) < \mathbf{w}_{b_{\inf}}(i) < \mathbf{u}_b(i)\}; \\
UP &= \{i | \mathbf{u}_b(i) = \mathbf{w}_{b_{\inf}}(i)\}.
\end{align*}
\]

3. Update \( D, k, f \) into \( D_{IN}, k_{IN}, f_{IN} \) by replacing the \( i^{th} \) row with \( \mathbf{e}_i \) for all \( i \in DOWN \) and \( i \in UP \), where \( \mathbf{e}_i = [0, \ldots, 0, 1, 0, \ldots, 0] \), the \( i^{th} \) element is 1, and others are 0.

4. Using the updated \( D_{IN}, k_{IN}, f_{IN} \), solve for an updated \( y \) according to equation (A-7). Compute an updated \( d\mathcal{L} \) using the updated \( y \):

\[
d\mathcal{L} = \lambda_b f - D \times (D_{IN}^{-1}k_{IN} + \lambda_b D_{IN}^{-1}f_{IN}) \equiv dLa + dLb \times \lambda_b,
\]

where \( dLa = -D \times D_{IN}^{-1}k_{IN} \), and \( dLb \equiv f - D \times D_{IN}^{-1}f_{IN} \).

5. Now decrease risk tolerance. This step finds the next value of \( \lambda_b \) that is smaller than the previous one, at which a variable in \( \mathbf{w}_{b_{\inf}} \) must change status, i.e., the next biggest critical value of risk tolerance. Alternatively, for any risk tolerance between this critical value and the previous one, the efficient portfolio set remains unchanged. The status of the portfolio weights can change in four different ways:

- \( i \in IN \) moves to \( UP \) when \( D_{IN}^{-1}f_{IN} < 0 \). Then the critical value of risk tolerance for asset \( i \), denoted by \( \text{crt}(i) \), is reached when \( y(i) = \mathbf{u}_b(i) \) – that is, when:

\[
\text{crt}(i) = \frac{\mathbf{u}_b(i) - D_{IN}^{-1}k_{IN}(i)}{D_{IN}^{-1}f_{IN}(i)}
\]

- \( i \in IN \) moves to \( DOWN \) when \( D_{IN}^{-1}f_{IN} > 0 \). The critical value is reached when
\[ y(i) = \ell b_b(i) \] – that is, when:

\[ \text{crt}(i) = \frac{\ell b_b(i) - D^{-1}_{IN} k_{IN}(i)}{D^{-1}_{IN} f_{IN}(i)} \]

- \( i \in \text{DOWN} \) moves to \( \text{IN} \) when \( d\ell b(i) < 0 \). The critical value is reached when \( d\mathcal{L}(i) = 0 \) – that is, when:

\[ \text{crt}(i) = -d\ell a(i)/d\ell b(i) \]

- \( i \in \text{UP} \) moves to \( \text{IN} \) when \( d\ell b(i) > 0 \). The critical value is reached when \( d\mathcal{L}(i) = 0 \) – that is, when:

\[ \text{crt}(i) = -d\ell a(i)/d\ell b(i) \]

For an asset weight that does not change status, assign \( \text{crt}(i) = 0 \).

6. Find the maximum critical value in the \((N + 1) \times 1\) vector \( \text{crt} \). The asset weight of the corresponding index \( i \) of the maximum critical value will change status according to the rule in step 5. Update the sets \( \text{DOWN}, \text{IN}, \) and \( \text{UP} \).

7. Repeat steps 3 - 6 until the last critical value of risk tolerance hits zero or is negative.

Following the algorithm, we compute corner portfolios and the associated critical values of risk tolerance. Then, according to Lemma 2.1, we can trace out the entire efficient frontier. Below I show a simple numerical example that has fixed expected returns and risks of bank debts.

**Example B.1. (Critical Line Method to Compute Efficient Frontier)**

Consider a case in which there are two banks and one firm asset. Assume that the boundary for a bank debt is no greater than two times its bank capital. Allow no short selling. Let bank capital be \( e = [100, 200]^{\top} \). Let the expected returns on bank debts be \( \mu = [2.8, 6.3, 10.8]^{\top} \). The variance-covariance matrix is given by

\[
V = \begin{pmatrix}
1.0000 & 2.9600 & 2.3100 \\
2.9600 & 54.7600 & 39.8860 \\
2.3100 & 39.8860 & 237.1600
\end{pmatrix}.
\]

Panel (a) and (b) in Figure A-1 display the efficient portfolio sets of the two banks against their
risk tolerances. Each kink represents a critical value of risk tolerance associated with a corner portfolio. When risk tolerances are large enough, banks’ optimal portfolio choices will be the same as when they are risk neutral; that is, when risk tolerance is infinity. Panel (c) and (d) in Figure A-1 display efficient frontiers in the $\mu$-$\sigma$ diagram. Because of the boundaries on portfolio weights, efficient frontiers are kinked as well.

![Efficient Portfolio Set of Bank 1](image1)

![Efficient Portfolio Set of Bank 2](image2)

![Efficient Frontier of Bank 1](image3)

![Efficient Frontier of Bank 2](image4)

Figure A-1: Efficient Frontiers: The Critical Line Method

Appendix C  Proof of Proposition 2.2

Recall the Kuhn-Tucker conditions (A-3) - (A-5) for any bank $b$’s optimal portfolio selection problem 4:

$$\frac{dL_b}{dw_b} = \lambda_b \mu - 2V w_b - \gamma_b 1 = 0,$$
\[ \mathbf{1}^\top w_b = 1, \]
\[ lb_b \leq w_b \leq ub_b. \]

The conditions can be rewritten as:

\[ \lambda_b \begin{pmatrix} \mu_B \\ \mu_T \end{pmatrix} - 2 \begin{pmatrix} V_B & 0 \\ 0 & V_T \end{pmatrix} \begin{pmatrix} w_{Bb} \\ w_{Tb} \end{pmatrix} - \gamma_b \mathbf{1}_{(N+K) \times 1} = 0, \quad (A-8) \]

\[ \mathbf{1}_{(N+K) \times 1}^\top \begin{pmatrix} w_{Bb} \\ w_{Tb} \end{pmatrix} = 1, \quad (A-9) \]

\[ \begin{pmatrix} lb_{Bb} \\ lb_{Tb} \end{pmatrix} \leq \begin{pmatrix} w_{Bb} \\ w_{Tb} \end{pmatrix} \leq \begin{pmatrix} ub_{Bb} \\ ub_{Tb} \end{pmatrix}, \quad (A-10) \]

where subscription \( B \) represents the partition associated with interbank debts for all variables (that is, the first \( N \) rows of each variable in matrix form). Subscription \( T \) represents the partition associated with non-financial firm assets for all variables; in other words, the \( N+1 \) to the \( N+K \) rows of each variable in matrix form. \( \mathbf{1} \) is an \( N + K \) by 1 vector of ones.

Condition (A-8) is equivalent to

\[ \begin{pmatrix} \lambda_b \mu_B \\ \lambda_b \mu_T \end{pmatrix} - 2 \begin{pmatrix} V_B & 0 \\ 0 & V_T \end{pmatrix} \begin{pmatrix} w_{Bb} \\ w_{Tb} \end{pmatrix} - \gamma_b \mathbf{1}_{N \times 1} = 0. \quad (A-11) \]

Condition (A-11) is a linear system that can be separated as the following two linear systems:

\[ 2V_B w_{Bb} = \lambda_b \mu_B - \gamma_b \mathbf{1}_{N \times 1}, \quad (A-12) \]

\[ 2V_T w_{Tb} = \lambda_b \mu_T - \gamma_b \mathbf{1}_{K \times 1}. \quad (A-13) \]

Solve for \( w_{Tb} \) in equation (A-13):

\[ w_{Tb} = \frac{1}{2} V_T^{-1} [\lambda_b \mu_T - \gamma_b \mathbf{1}_{K \times 1}]. \quad (A-14) \]

Note that \( V_T, \mu_T \) and \( \lambda_b \) are fixed; \( \gamma_b \) is endogenous. When \( \gamma_b \) varies when we solve for optimal portfolio weights, every equation in the linear system (A-13) faces the identical change on the
right-hand side of the equation. However, the changes on the right-hand side are not proportional. In other words, take the multiplicity of matrix $V_T$ as a projection of vector $w_{Tb}$ to a vector defined as the right-hand side of (A-13). This vector changes as $\gamma_b$ varies. According to conditions (A-12), the value of $\gamma_b$ is relevant to bank $b$’s choice of interbank debts $w_{Bb}$. Therefore, $w_{Bb}$, $w_{Tb}$ and $\lambda_b$ interact to determine the optimal portfolio weights of bank $b$ even when we assume that bank debt returns and non-financial asset returns are uncorrelated.

Furthermore, consider a special case where all banks face only one firm asset $t$, or one mutual fund. Then equations (A-13) become

$$2\sigma_t^2 w_{tb} + \gamma_b = \lambda_b \mu_t. \tag{A-15}$$

Rewrite equation (A-12) as

$$2V_B w_{Bb} + \gamma_b 1_{N \times 1} = \lambda_b \mu_B,$$

and divided by equation (A-15) for $\lambda_b > 0$. We have:

$$\frac{1}{2\sigma_t^2 w_{tb} + \gamma_b} \times (2V_B w_{Bb} + \gamma_b 1_{N \times 1}) = \frac{\mu_B}{\mu_t}. \tag{A-16}$$

Since short-selling is forbidden in the model, $w_{tb}$ is non-negative. In addition, $\gamma_b$ is always non-negative as a Lagrange multiplier. Thus, portfolio weights on bank debts $w_{Bb}$ relative to the portfolio weight on the non-financial firm $w_{tb}$ are positively dependent on the relative expected returns of the bank debts with respect to that of the firm asset, that is, $\frac{\mu_B}{\mu_t}$.

**Appendix D Algorithm to Compute Equilibrium in the Banking Sector**

Given parameter values, I use a grid search method to find equilibrium of the interbank debt network.
Algorithm 1 Equilibrium in the Banking Sector

1: procedure Trace Out Demand and Supply Curves of Bank Debts
2:   input parameter values in problem (4)
3:   define a range for the expected return of the factor $\mu_f$ to adjust, and number of grid
search points
4:   compute vectors of expected returns of bank debt according to equations (6) - (8) as $\mu_f$
adjusts
5:   for each grid search do
6:     for each bank $b \in B$ do
7:       trace out efficient frontier
8:       using the bank’s risk tolerance, locate the optimal portfolio on the frontier
9:     end for
10:  end for
11:  for each bank debt $b \in B$ do
12:    compute the dollar amount of the demand and supply of the bank debt for all grids
13:  end for
14: end procedure
15: procedure Find Equilibrium Satisfying Market Clearing Conditions
16:   for each bank debt $b \in B$ do
17:     find the intersection(s) of its demand and supply curve and its corresponding $\mu_{fb}$
18:   end for
19: loop
20: if there is a bank debt that does not have coincident demand and supply then
21:   there is no equilibrium interbank debt network, end the loop
22: else
23:   look for common expected return(s) of the factor $\mu_{fb}$ that clear every bank debt market
24:   if a common expected return of the factor does not exist then
25:     there is no equilibrium interbank debt network, end the loop
26:   else
27:     the common value(s) $\mu_{fb}^*$ will be the equilibrium
28:     compute equilibrium expected returns for all bank debts according to equations (6) - (8)
29:     for each bank $b \in B$ do
30:       use the critical line method again to compute the equilibrium optimal portfolio weights, which determine the interbank debt network
31:     end for
32:   end if
33: end if
34: end loop
35: end procedure
Appendix E  Robustness of Comparative Statics

Judging from the comparative statics of increasing banks’ risk tolerances, banks tend to form more links or increase interbank lending activities. I conduct a robustness check of this result by varying parameter values. Under different parameter values, banks will form distinct network structures. In different network structures and parameter values, the above result holds. In this section, I display two examples. In the first, banks form a complete network. In the second, banks form an incomplete network, but as their risk tolerances increase the bank network becomes complete.

Example E.1. (Complete Network)

Consider a banking sector with four banks that are of the same size. There is one non-financial firm. \( e = [10, 10, 10, 10]^\top \). Set the bank debt upper boundary to be 100 times its bank capital. Let the expected return of the factor \( \mu_f \) vary in a range \([0.01\%, 50\%]\) and standard deviation of the factor be \( \sigma_f = 18\% \). In the factor model, expected returns of bank debts are assumed to follow linear relationships:

\[
\begin{align*}
\mu_1 &= 0.015 + 1.2\mu_f, \\
\mu_2 &= 0.012 + 1.1\mu_f, \\
\mu_3 &= 0.011 + \mu_f, \\
\mu_4 &= 0.01 + 0.9\mu_f.
\end{align*}
\]

Variances of the residuals are \( \sigma^2_e = [0.02, 0.018, 0.01, 0.005]^\top \). The firm asset has an expected return of 2.5\% and standard deviation of 25.8457\%. Parameter values indicate that there no asset dominates the others. Increase: the risk tolerance of bank 1 from 41.86 to 909.539; the risk tolerance of bank 2 from 6.505 to 9.837; the risk tolerance of bank 3 from 4.85 to 5.26; and the risk tolerance of bank 4 from 0.01 to 0.42.

As shown in Figure A-2a, banks in equilibrium form a complete network\(^{21}\). In this network bank 1 is a pure borrower and bank 4 is a pure lender. Equilibrium expected returns on bank debts increase as risk tolerances increase because banks tend to invest more in the debts of other banks. In Figure A-2, panel (c) through (f) show the portfolio weights of banks that are

\(^{21}\)Arrows in the network plot have the same direction as cash flows at time 0. For example, an arrow from bank 4 to bank 2 means that bank 4 lends to bank 2.
in equilibrium. As risk tolerances increase, banks 1, 2, and 3 take on higher leverage as debtors; this is especially so in the case of bank 1, which borrows more than its bank capital. Banks 2, 3, and 4 lend more in the banking sector as creditors but they invest less in the firm asset. Banks become more connected as their risk tolerances increase.

(a) Interbank Lending Network
(b) Equilibrium Expected Bank Debt Returns

c) Bank 1 Portfolio
d) Bank 2 Portfolio

e) Bank 3 Portfolio
f) Bank 4 Portfolio

Figure A-2: Comparative Statics of Four Banks: Complete Network
Example E.2. (Incomplete Network Evolves to Complete Network)

As was done in Example 3.1, assume that $e = [10, 10, 10, 10]^\top$. Set the upper boundary for a bank debt to be 100 times its bank capital. Let the expected return of the factor $\mu_f$ vary in a range $[0.01\%, 50\%]$ and the standard deviation of the factor be $\sigma_f = 18\%$. In the factor model, the expected returns of bank debts are assumed to follow linear relationships:

\[
\begin{align*}
\mu_1 &= 0.015 + 1.2\mu_f, \quad (A-21) \\
\mu_2 &= 0.012 + 1.1\mu_f, \quad (A-22) \\
\mu_3 &= 0.011 + \mu_f, \quad (A-23) \\
\mu_4 &= 0.01 + 0.9\mu_f. \quad (A-24)
\end{align*}
\]

Variances of the residuals are $\sigma^2_\epsilon = [0.015, 0.012, 0.01, 0.008]^\top$. The firm asset has an expected return of 2.5% and a standard deviation of 7%. Increase: the risk tolerance of bank 1 from 3.123 to 49.7; the risk tolerance of bank 2 from 0.6492 to 1.1537; the risk tolerance of bank 3 from 0.01 to 0.09; and the risk tolerance of bank 4 from 0.77 to 0.85.

As shown in Figure A-3a, banks in equilibrium form an incomplete network when risk tolerances rise up to $[9.12, 0.9245, 0.07, 0.83]$. When risk tolerances go above $[13.44, 1, 0.08, 0.84]$, bank 3 becomes a creditor of bank 1 and the network becomes complete. As risk tolerances increase, banks 1 and 2 take on higher leverage. Banks 2 and 4 lend more to bank 1. Bank 3 lends more to bank 2, but it lends less to bank 4 when it forms a new link with bank 1. However, the total credit that bank 3 provides in the banking sector increases. Changes in portfolio weights fall into a relatively small range because bank risk tolerances vary in only a small range.
Figure A-3: Comparative Statics of Four Banks: Incomplete to Complete Network

(a) Interbank Lending Network

(b) Equilibrium Expected Bank Debt Returns

(c) Bank 1 Portfolio

(d) Bank 2 Portfolio

(e) Bank 3 Portfolio

(f) Bank 4 Portfolio
Appendix F  Fictitious Default Algorithm

The fictitious default algorithm was developed by Eisenberg and Noe (2001). The main steps are as follows:

1. Let $D(p) = \{ b : \Phi(p)b < L_b, b \in B \}$ be the set of default bank.

2. For given networks of inter-bank debt and investment in firm assets $A$, $I$, $M$, and $L$, as well as endowments $e$, assume all banks pay full obligations, $p^0 = L$.

3. If under $p^0$ all obligations are satisfied, then terminate the algorithm; there is no default in the banking system. If some banks default even when all other banks pay, then these defaults are "first-order" defaults and these banks belong to $D(p)$. Update $p$ assuming only first-order defaults occur.

4. Under the new $p$, if only first-order defaults occur, then terminate the algorithm. If second-order defaults occur, then update $p$ again and check for higher order of defaults.

5. Repeat until there is no further default.

When the algorithm starts the total default amount in the entire banking system is zero. In each round of iteration the total default amount strictly increases; thus, the algorithm will not fall into infinite loops. The iteration will stop when the total default amount stops increasing or when it reaches the maximum possible level – that is, when all banks pay zero. Therefore, the algorithm ends in finite iterations, and through rounds of defaults we are able to track how defaults spread in the banking system.
References


Popov, Alexander and Gregory F Udell (2012). “Cross-Border Banking, Credit Access, and the


Risk Centre Special Paper Series, London School of Economics and Political Science*. 