**CAEPR Working Paper
#2016-002**

Nonparametric Identification of Dynamic Games with Multiple Equilibria and Unobserved Heterogeneity

**Ruli Xiao
Indiana University**

March 7, 2016

This paper can be downloaded without charge from the Social Science Research Network electronic library at http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2757272

The Center for Applied Economics and Policy Research resides in the Department of Economics at Indiana University Bloomington. CAEPR can be found on the Internet at: <http://www.indiana.edu/~caepr>. CAEPR can be reached via email at caepr@indiana.edu or via phone at 812-855-4050.

©2016 by Ruli Xiao. All rights reserved. Short sections of text, not to exceed two paragraphs, may be quoted without explicit permission provided that full credit, including © notice, is given to the source.

Nonparametric Identification of Dynamic Games with Multiple Equilibria and Unobserved Heterogeneity *

Ruli Xiao

Indiana University

March 7, 2016

Abstract

This paper provides sufficient conditions for non-parametrically identifying dynamic games with incomplete information, allowing for both multiple equilibria and unobserved heterogeneity. The identification proceeds in two steps. The first step mainly involves identifying the equilibrium conditional choice probabilities and the state transitions using results developed in the measurement error literature. The existing measurement error literature relies on monotonicity assumptions to determine the order of the latent types. This paper, in contrast, explores the identification structure to match the order, which is important for identifying the payoff primitives. The second step follows existing literature to identify the payoff parameters based on the equilibrium conditions with exclusion restrictions. Multiple equilibria and unobserved heterogeneity can be distinguished through comparison of payoff primitives.

JEL Classification: C14

Keywords: Multiple equilibria, Unobserved heterogeneity, Discrete games, Dynamic games, Non-parametric identification

*I am deeply indebted to Yingyao Hu for his generous support and guidance. I benefited greatly from the comments of Yuya Sasaki and Richard Spady. I thank Victor Aguirregabiria, Jorge Balat, Yao Luo, Matt Shum, Andrew Sweeting, Yuya Takahashi, Nathan Yang, and the seminar participants at the Johns Hopkins University and University of Toronto for their helpful comments. All errors are mine. Comments are welcome.

1 Introduction

There are two outstanding challenges involved in the estimation of dynamic discrete games: (1) the presence of unobserved heterogeneity at the firm or market levels; (2) the indeterminate of the number of equilibria. Existing literature attempts to tackle one or the other of these two problems but not both together.¹ Allowing for both unobserved market-level heterogeneity and multiple equilibria, this paper provides sufficient conditions for non-parametrically identifying dynamic games with incomplete information. Specifically, these conditions enables identification of all aspects of the game, including the cardinality and the marginal distribution of the unobserved factor, the number of equilibria, the equilibrium selection probability, the equilibrium conditional choice probabilities (CCPs), the state transitions, and the payoff primitives.

The identification proceeds in the following steps. Firstly, I identify the cardinality of the aggregate latent variable, which combines unobserved heterogeneity and multiple equilibria. Secondly, I identify the equilibrium CCPs and state transitions by modifying results developed in measurement error literature, such as Hu and Shum (2012). Thirdly, I identify the payoff functions with exclusion restrictions, as in Bajari et al. (2009). Lastly, I distinguish between unobserved heterogeneity and multiple equilibria by testing whether the payoff primitives are the same or not. Specifically, multiple equilibria are associated with the same payoff functions, while unobserved-market types are associated with different levels of payoff primitives. Similar idea has been explored in Aguirregabiria and Mira (2015) for static games.

Unlike the measurement error literature, this paper does not impose any monotonicity conditions. The monotonicity condition is required to determine the ordering of the unobserved type because the latent components are identified only up to permutation. Such monotonicity conditions are easily violated in games without multiple equilibria. For games with multiple equilibria, it becomes more difficult for similar monotonicity conditions to hold. To address the ordering problem, this paper explores the identification structure to match the order for different values of the unobservables without

¹Aguirregabiria and Mira (2007) allows for unobserved heterogeneity, while Pesendorfer and Takahashi (2012) tests for the existence of multiple equilibria in dynamic games. See Nevo and Aguirregabiria (2010) for a survey.

determining the exact ordering. That is, the unobserved factor is identified up to permutation, but the ordering is the same across all observables. This matched order enables identification of the payoff primitives, but only up to permutation. Relaxing the monotonicity assumption is important in practice, because it does not hold trivially even in single-agent discrete choice models.

It is important and complicated to address unobserved heterogeneity in identification and estimation of games. Ignoring the presence of unobserved heterogeneity is unrealistic for some empirical IO applications and also problematic in explaining micro data. Not accounting for potentially unobserved heterogeneity may lead to significant biases in parameter estimates,² and thus creates misunderstandings of strategic interaction between firms. Meanwhile, investigating the identification with unobserved heterogeneity is important for its widely inclusion in empirical estimations. Identification may follow the results developed in the finite mixture/measurement error literatures (Kasahara and Shimotsu (2009) and Hu and Shum (2012)). It is more difficult, however, to tackle unobserved heterogeneity in games than in discrete choice models, because of the possible coexistence of multiple equilibria.

Indeed, the presence of multiple equilibria is a prevalent feature in dynamic games. Identification of games with multiple equilibria is not well-understood, even though the presence of multiple equilibria does not necessarily preclude the identification (Jovanovic (1989)). For instance, focusing on one single market enables identification and consistent estimation of the payoff primitives since Markov Perfect Equilibrium implicitly assumes a single equilibrium is employed (Pesendorfer and Schmidt-Dengler (2008)). Relying on a long span time series, however, is often not feasible in practice sometimes due to the limited availability of data. A common estimation approach that may be used instead is to use cross market variation. For the validity of this approach, the pooled data has to be generated from the same equilibrium, which rarely has empirical evidence. Imposing such a restriction may result in the mis-specification and inconsistent estimation of payoff primitives.

To the best of my knowledge, this paper is the first to provide rigorous identification results for dynamic games, while incorporating unobserved heterogeneity and multiple equilibria. The identification

²For instance, in the empirical application in Aguirregabiria and Mira (2007), the estimation without unobserved market heterogeneity implies estimates of strategic interaction between firms (that is, competition effects) that are close to zero or even have a sign opposite to that expected under competition. While including unobserved heterogeneity in the models results in estimates that show significant and strong competition effects.

results presented in this paper are of real practical importance. With fully understanding of the conditions under which the underlying data generating process can be achieved non-parametrically, one will be more comfortable about the estimation results regardless its functional assumptions. Even though some existing literature considers estimation of dynamic games allowing for unobserved heterogeneity, there is no rigorous discussion about the identification (See Aguirregabiria and Mira (2007), Bajari et al. (2007), and Arcidiacono and Miller (2011)).

This paper also relates to the literature on games with multiple equilibria.³ Otsu et al. (2015) propose several statistical tests for finite state Markov games, in order to examine whether the data can be pooled for estimation. Xiao (2014) provides identification results for static games with multiple equilibria. Aguirregabiria and Mira (2015) consider identification of games allowing for multiple equilibria and unobserved heterogeneity in static settings, and focus on distinguishing between multiple equilibria and unobserved heterogeneity.

The remainder of the paper is organized as follows. I begin by describing the game framework in section 2. Section 3 provides the nonparametric identification results. Section 4 concludes. The Appendix contains the proofs.

2 Dynamic Games

Consider a model of discrete-time, infinite-horizon games with N players.⁴ At the beginning of each period $t(t \in \{0, 1, \dots, \infty\})$, the players simultaneously determine which action to take. Let a_{it} and a_t denote player i 's action and an action profile, respectively, in period t , i.e., $a_{it} \in A_i = \{0, 1, \dots, K\}$ and $a_t = \{a_{1t}, \dots, a_{Nt}\}$. Before making a decision, player i observes a vector of state variables s_t and a vector of action-specific private payoff shocks $\epsilon_{it} = (\epsilon_{it}(a_{it} = 0), \dots, \epsilon_{it}(a_{it} = K))$. Let ϵ_t represent the private information for all players, i.e., $\epsilon_t \equiv (\epsilon_{1t}, \dots, \epsilon_{Nt})$.

³See De Paula (2012) for a survey of the recent literature on the econometric analysis of games with multiplicity. See also Sweeting (2013), Ciliberto and Tamer (2009) for bound identification, Bajari et al. (2010), De Paula and Tang (2012) for identifying the sign of the strategic interaction term using multiple equilibria

⁴See a similar framework used in Ericson and Pakes (1995), Aguirregabiria and Mira (2007), Pakes et al. (2007), Bajari et al. (2007), Pesendorfer and Schmidt-Dengler (2008), and Pesendorfer and Schmidt-Dengler (2010).

The state variable, s_t , consists of players' previous actions along with market size and individual firm characteristics, i.e., $s_t = (x_t, a_{t-1})$, where x_t includes all characteristics except previous actions. In empirical applications, the previous action enters a player's payoff function directly. For example, in Sweeting (2013), the format that music stations choose to air in a given period depends on the format they aired in the previous period, due to the switching cost. Some dynamic games are to analyze firms strategic interaction regarding entry or exit, in which previous actions affect current payoffs (see also Igami and Yang (2015)). The identification method proposed in this paper is also applicable to games that previous actions do not affect firms' payoffs.

To characterize the equilibrium, I first introduce several assumptions imposed in the existing literature.

Assumption 1. (*Conditional Independence*) *The payoff shocks ϵ_{it} are independent across actions and players and over time. Moreover, the payoff shocks ϵ_{it} have support of R^{K+1} .*

The assumption of independence of payoff shocks is to facilitate tractability. The correlation of payoff shocks calls for a model of learning that captures the evolution of a player's belief over opponents' payoff shocks with knowledge of their past actions, which greatly increases the size of the state variable. The state is assumed to be discrete and finite; that is $s \in S$, where S is the support of the state s .

Assumption 2. (*State Evolution*) *The state transition is described by a probability density function $g : S \times A \times S \rightarrow [0, 1]$, where $g(s_{t+1}|a_t, s_t)$ is the probability that state s_{t+1} is reached given the previous state s_t , and the action profile a_t .*

To simplify the model, this assumption rules out the scenario in which all past history affects the evolution of the state variable. Naturally, $\sum_{s'} g(s_{t+1} = s'|a_t, s_t) = 1$.

Assumption 3. (*Additive Separability*) *The payoff for player i from choosing action a_{it} while her rivals choose actions a_{-it} in period t is assumed to be additively separable, as follows:*

$$u_i(a_t, s_t, \epsilon_{it}) = \pi_i(a_{it}, a_{-it}, s_t) + \epsilon_{it}(a_{it}).$$

This paper considers only Markov Perfect Equilibria(MPE), in which players' actions only depend only on the current state variable; that is, historical information is irrelevant given the most current state. With this Markovian property, the game is stationary so I suppress t for ease of notation. Let $\delta_i(s, \epsilon_i) : S \times R^{K+1} \rightarrow A$ be player i 's strategy; it prescribes an action for player i given the available public and own private information, and let $\delta = \{\delta_i(s, \epsilon_i)\}$ be a strategy profile. I define conditional choice probabilities (CCPs) $p_i(a_i|s)$ as the probability that player i chooses action a_i given state s . Thus,

$$p_i(a_i|s) \equiv \Pr(\delta_i(s, \epsilon_i) = a_i) = \int_{\epsilon_i} I(\delta_i(s, \epsilon_i) = a_i) f(\epsilon_i) d\epsilon_i,$$

where $I(\cdot)$ is the indicator function. There is a one-to-one mapping between the set of CCPs $p = \{p_i(a_i|x)\}_i$ and the strategy profile δ . Let $W_i(s, \epsilon_i; \delta)$ be player i 's value function given state s , own private information ϵ_i , and other firms following their strategies in δ . Thus,

$$W_i(s, \epsilon_i; \delta) = \max_{a_i \in A_i} \{ \Pi_i(a_i, s) + \epsilon_i(a_i) + \beta \int \sum_{a_{-i}} W_i(s', \epsilon'_i; \delta) g(s'|s, a_i, a_{-i}) p_{-i}(a_{-i}|s) f(\epsilon'_i) d\epsilon'_i ds' \},$$

where $\Pi_i(a_i, s) = \sum_{a_{-i}} \pi_i(a_i, a_{-i}, s) p_{-i}(a_{-i}|s)$ is firm i 's current expected profit with action a_i and the other firms behaving according to their respective strategies in δ . I also define respectively the ex ante value function and the choice specific value function for player i given that the other firms behave according to their strategies in δ , as follows:

$$\begin{aligned} V_i(s) &= E_{\epsilon_i} W_i(s, \epsilon_i; \delta) \\ V_i(a_i, s) &= \Pi_i(a_i, s) + \beta E W_i(s', \epsilon'_i; p) = \Pi_i(a_i, s) + \beta \sum_{s'} V_i(s') g(s'|a_i, s). \end{aligned} \quad (1)$$

With above definition and notation, the MPE can be characterized as follows:

Definition 1. (MPE) An MPE is a set of strategy functions δ^* such that for any firm i and for any $(s, \epsilon_i) \in S \times R^{K+1}$,

$$\delta_i^*(s, \epsilon_i) = \operatorname{argmax}_{a_i \in A_i} \{ v_i(a_i, s) + \epsilon_i(a_i) \}.$$

For any set of strategies δ , in equilibrium or not, the payoff components depend on players' strategies

only through CCPs associated with strategy δ .⁵ With CCPs, the equilibrium conditions become

$$p_i(a_i = k, s) = \Pr(V_i(a_i = k, s) + \epsilon_i(a_i = k) \geq V_i(a_i = j, s) + \epsilon_i(a_i = j), \forall j).$$

Under the assumption that ϵ_i is Type I extreme value distributed private shocks, we have the following ex ante value function:

$$\begin{aligned} V_i(s) &= E_{\epsilon_i} \max_{a_i} (V_i(a_i, s) + \epsilon_i(a_i)) \equiv \sum_{a_i \in A_i} \Pi_i(a_i, s) p_i(a_i | s) + \beta \sum_{a_i} \sum_{s'} V_i(s') (g(s' | a_i, s) + e_i^P(a_i, s)) p_i(a_i | s) \\ &= \Psi_i^V(s, V, P, \pi), \end{aligned} \tag{2}$$

where $e_i^P(a_i, s) = \text{Euler's constant} - \log(p_i(a_i | s))$, $V = \{V_i(x)\}_{i \in N, s \in S}$ collects all ex ante value functions into a vector, and $P = \{p_i(a_i | s)\}_{a_i \in A_i, i \in N, s \in S}$. In addition, the equilibrium conditions become

$$p_i(a_i = k, s) = \frac{\exp(V_i(a_i = k, s))}{\sum_j \exp(V_i(a_i = j, s))} \equiv \Psi_i^P(a_i = k | s, V, P, \pi). \tag{3}$$

The above two equations are satisfied for each individual, every action, and all possible values of the state variable. To further simplify notation, define $\Psi^P(V, P, \pi) = \Psi_i^P(s, V, P, \pi)_{a_i \in A_i, i \in N, s \in S}$ and $\Psi^V(V, P, \pi) = \Psi_i^V(s, V, P, \pi)_{i \in N, s \in S}$. Thus, an MPE in the probability space P can be characterized as a solution to the following system of non-linear equations

$$\Omega(\pi, g, \Psi) = \{(P, V) | V = \Psi^V(V, P, \pi) \ \& \ P = \Psi^P(V, P, \pi)\},$$

where the equilibrium mapping Ψ is determined by the distribution of payoff shocks (Egesdal et al. (2015)). Note that multiple equilibria could exist.

⁵MPE defined in the strategy space is equivalent to that defined in the probability space (Milgrom and Weber (1985)).

3 Non-parametric Identification

The econometrician observes firms' action profiles a_t^m up to T periods in market m for $m = 1, \dots, M$, and the market and individual characteristics x_t^m in each period. Here the characteristics are assumed to be discrete and finite, i.e., $x \in X = \{X^1, \dots, X^H\}$. Data can be summarized as

$$\{a_t^m, x_t^m, t = 1, \dots, T, m = 1, \dots, M\}.$$

3.1 Data Generating Process

There are two types of latent factors, including unobserved market types and multiple equilibria. The time-variant unobserved market-level heterogeneity, which is denoted as η_t , is payoff-relevant. Assume that it is finite; that is, $\eta_t \in \Psi \equiv \{\eta^1, \dots, \eta^L\}$. Multiple equilibria, on the other hand, are payoff-irrelevant. Note that the number of equilibria is finite and discrete.⁶ I denote the equilibrium and the equilibrium set as e^* and $\omega(\pi, g, \Psi)$, respectively.

This paper assumes exogenous equilibrium selection process, such as by nature or some outside mechanisms, following the existing literature. The determinant of equilibrium is characterized by a probability distribution denoted as $p^e(\pi, g, \Psi) \equiv \{\Pr(e^*), e^* \in \omega(\pi, g, \Psi)\}$. Furthermore, I allow some equilibria to be selected with a zero probability, i.e., $\Pr(e^*) = 0$ for some e^* . I consider an equilibrium with a positive selection probability as an active equilibrium, and denote the active equilibrium set as $\omega^a(\pi, g, \Psi) = \{e^* : \Pr(e^*) > 0 \text{ \& } e^* \in \omega(\pi, g, \Psi)\}$. The number of the active equilibria, which is denoted as $Q = \#\{e^*, \Pr(e^*) > 0 \text{ \& } e^* \in \omega(\pi, g, \Psi)\}$, may be different from the total number of equilibria. Modeling the equilibrium selection process is too complicated and out of the scope of this paper.

The model primitives then can be summarized as

$$\{\pi, g, \Omega(\pi, g, \Psi), \omega(\pi, g, \Psi), p^e(\pi, g, \Psi)\}.$$

⁶As shown in Haller and Lagunoff (2000), stochastic dynamic games also have a finite number of equilibria.

3.2 First Step Identification

The overall identification proceeds in two steps. First I show how to identify the equilibrium CCPs and state transition using the joint distribution of observables. I then show how to identify payoff primitives following the existing literature.

Identifying the equilibrium CCPs and state transition requires the following conditions.

Assumption 4. *The market observable x_t and unobservable η_t evolve according to the following rules:*

$$(i). \Pr(\eta_t | x_{t-1}, \eta_{t-1}, a_{t-1}, \Omega_{<t-1}) = \Pr(\eta_t | \eta_{t-1}, x_{t-1}, a_{t-1}),$$

$$(ii). \Pr(x_t | \eta_t, x_{t-1}, \eta_{t-1}, a_{t-1}, \Omega_{<t-1}) = \Pr(x_t | \eta_t, x_{t-1}, a_{t-1}),$$

where $\Omega_{<t-1} \equiv \{x_{t-2}, \eta_{t-2}, a_{t-2}, \dots, x_1, \eta_1, a_1\}$, the history up to (but not including) $t - 1$.

Assumption 1(ii) indicates "limited feedback", which rules out direct feedback from the previous unobservable η_{t-1} , on the current observable x_t , but allows indirect effect of η_{t-1} through x_{t-1} and a_{t-1} . Implicitly, this evolution process imposes a timing restriction on the game characteristics, i.e., the unobserved characteristics η_t being realized before the observed characteristics x_t . As a result, x_t depends on η_t instead of η_{t-1} . However, this limited feedback assumption is less restrictive than the assumption made in many applied settings, where the observable x_t evolves independently from the unobservable η_t of any periods, so that the state transition of observables can be estimated directly from the data. This assumption, however, does rule out the scenario in which the alternative timing occurs. The limited feedback assumption is trivial when the unobserved market type does not vary overtime.

Lemma 1. *In a given market, observables and unobservables satisfy the following properties, and the joint distribution of observables satisfy the following representations:*

$$(i). \{w_t, \eta_t\} \equiv \{a_t, x_t, \eta_t\} \text{ follows a stationary first-order Markov process,}$$

$$(ii). \Pr(w_{t+2}, w_{t+1}, w_t) = \sum_{\eta_{t+1}} \Pr(w_{t+2} | w_{t+1}, \eta_{t+1}) \Pr(w_{t+1}, w_t | \eta_{t+1}) \Pr(\eta_{t+1}),$$

$$(iii). \Pr(w_{t+3}, w_{t+2}, w_{t+1}, w_t) = \sum_{\eta_{t+2}} \Pr(w_{t+3} | w_{t+2}, \eta_{t+2}) \Pr(w_{t+2} | w_{t+1}, \eta_{t+2}) \Pr(w_{t+1}, w_t, \eta_{t+2}).$$

Proof See Appendix ■

Lemma 1 holds for any individual markets over time. In some empirical applications, researchers rely on data pooled from different markets for estimation. With pooling data, estimation is consistent if the data is generated by the same equilibrium or there is a unique equilibrium. When there are multiple equilibria involved, the reduced-form outcome distributions computed from the data represent a mixture of outcome distributions associated with different equilibria. Moreover, estimation becomes more complicated if there is unobserved heterogeneity.

To incorporate both latent variables, I create an overall latent variable, denote as $\tau : \Psi \times \omega \rightarrow \Upsilon$, where $\tau(\cdot)$ is a function aggregating the unobserved heterogeneity η and the equilibrium e^* . Since both η and e^* are discrete and finite, so does τ ; that is, $|\Psi| \times |\omega| = |\Upsilon| \equiv Q^\tau$, where $|\cdot|$ is the cardinality. Intuitively, τ provides overall information on both latent variables. For instance, suppose η_{t+1} represents the unobserved market demand $\{high, low\}$, and there are two equilibria $\{1, 2\}$. One example of the overall latent factor τ_{t+1} is: $\tau_{t+1} = 1$ (the market employs equilibrium 1, and the current consumer demand is *high*), $\tau_{t+1} = 2$ (the market employs equilibrium 1, and the current consumer demand is *low*), $\tau_{t+1} = 3$ (the market employs equilibrium 2, and the current consumer demand is *high*), and $\tau_{t+1} = 4$ (the market employs equilibrium 2, and the current consumer demand is *low*).

With three periods of data, the joint distributions of pooling data becomes

$$\begin{aligned}
\Pr(w_{t+2}, w_{t+1}, w_t) &= \sum_{e^* \in \omega} \Pr(w_{t+2}, w_{t+1}, w_t | e^*) \Pr(e^*) \\
&= \sum_{e^* \in \omega^a} \Pr(w_{t+2}, w_{t+1}, w_t | e^*) \Pr(e^*) \\
&= \sum_{e^* \in \omega^a} \sum_{\eta_{t+1}} \Pr(w_{t+2} | w_{t+1}, \eta_{t+1}, e^*) \Pr(w_{t+1}, w_t | \eta_{t+1}, e^*) \Pr(\eta_{t+1}, e^*) \\
&= \sum_{\tau_{t+1}} \Pr(w_{t+2} | w_{t+1}, \tau_{t+1}) \Pr(w_{t+1}, w_t | \tau_{t+1}) \Pr(\tau_{t+1}). \tag{4}
\end{aligned}$$

I introduce the following matrix notation:

$$\begin{aligned}
F_{w_{t+2}, \bar{w}_{t+1}, w_t} &\equiv [\Pr(w_{t+2} = k, \bar{w}_{t+1}, w_t = j)]_{k,j}, \\
A_{w_{t+2}|\bar{w}_{t+1}, \tau_{t+1}} &\equiv [\Pr(w_{t+2} = k | \bar{w}_{t+1}, \tau_{t+1} = q)]_{k,q}, \\
B_{\bar{w}_{t+1}, w_t|\tau_{t+1}} &\equiv [\Pr(\bar{w}_{t+1}, w_t = k | \tau_{t+1} = q)]_{q,k}, \\
D_{\tau_{t+1}} &\equiv \text{diag}[\Pr(\tau_{t+1} = 1) \dots \Pr(\tau_{t+1} = Q^\tau)].
\end{aligned}$$

Those matrices stack the distributions with all of the possible values of w_t and τ_{t+1} . In particular, matrix $F_{w_{t+2}, \bar{w}_{t+1}, w_t}$ consists of the entire joint distributions of w_{t+2} and w_t with fixing w_{t+1} . Matrix $D_{\tau_{t+1}}$ is a diagonal matrix with the marginal distribution of τ_{t+1} as the diagonal elements, while matrix $A_{w_{t+2}|\bar{w}_{t+1}, \tau_{t+1}}$ collects transition probabilities.

To aggregate all of the possible equations, I rewrite equation (4) into the following matrix representation:

$$F_{w_{t+2}, \bar{w}_{t+1}, w_t} = A_{w_{t+2}|\bar{w}_{t+1}, \tau_{t+1}} D_{\tau_{t+1}} A_{\bar{w}_{t+1}, w_t|\tau_{t+1}}.$$

This matrix expression provides information on the cardinality of the aggregate latent variable, which I summarize in the following lemma.⁷

Lemma 2. *The rank of the observed matrix $F_{w_{t+2}, \bar{w}_{t+1}, w_t}$ serves as a lower bound for the cardinality of the aggregated latent variable τ_t , i.e., $Q^\tau \geq \text{Rank}(F_{w_{t+2}, \bar{w}_{t+1}, w_t})$. Furthermore, the cardinality is identified, in particular, $Q^\tau = \text{Rank}(F_{w_{t+2}, \bar{w}_{t+1}, w_t})$ when the following conditions are satisfied:*

(1) $\|X\| \times \|A^n\| > Q^\tau$; (2) both matrices $A_{w_{t+2}|\bar{w}_{t+1}, \tau_{t+1}}$ and $A_{\bar{w}_{t+1}, w_t|\tau_{t+1}}$ have full column rank.

Proof See Appendix. ■

The first condition indicates that the measurement's cardinality needs to be greater than that of the latent variable. Secondly, the full rank condition indicates the requirement of enough variations in the

⁷The identification argument is similar to that of Kasahara and Shimotsu (2014).

conditional choice probability of different equilibria for disentangling CCPs for each equilibrium. Note that the full rank condition is required for only one value of w_{t+1} . In practice, we can check the rank of the matrix associated with different values of w_{t+1} , and the cardinality should be the highest rank.

Having identified the cardinality of the aggregate latent variable, I now proceed to identify the law of transition for both the observables and unobservables. The identification again relies on the joint distribution of the observables, but requires four periods of data. In this case, the joint distribution of the observables becomes:

$$\Pr(w_{t+3}, w_{t+2}, w_{t+1}, w_t) = \sum_{\tau_{t+2}} \Pr(w_{t+3}|w_{t+2}, \tau_{t+2}) \Pr(w_{t+2}|w_{t+1}, \tau_{t+2}) \Pr(w_{t+1}, w_t, \tau_{t+2}). \quad (5)$$

With the cardinality of the latent factor τ_{t+2} identified, I regroup the state space into a dimension of Q^τ , and denote the regrouped variable as $z_t = z(w_t)$, such that the corresponding matrix $F_{z_{t+2}, \bar{w}_{t+1}, z_t}$ is full rank. Fixing w_{t+2} and w_{t+1} , I rewrite equation (5) into the following matrix expression

$$F_{z_{t+3}, w_{t+2}, w_{t+1}, z_t} = A_{z_{t+3}|w_{t+2}, \tau_{t+2}} D_{w_{t+2}|w_{t+1}, \tau_{t+2}} B_{w_{t+1}, z_t, \tau_{t+2}}. \quad (6)$$

Identification of $\Pr(w_{t+3}|w_{t+2}, \tau_{t+2})$ is obtained by evaluating the joint distribution of four periods of data at four pairs of points (w_{t+2}, w_{t+1}) , $(\bar{w}_{t+2}, w_{t+1})$, $(w_{t+2}, \bar{w}_{t+1})$, $(\bar{w}_{t+2}, \bar{w}_{t+1})$. Matrix equations from each of these four pairs share one matrix in common. Canceling common matrices, I form an eigenvalue-eigenvector representation between the observed and unknown matrices in the following:

$$\begin{aligned} & F_{z_{t+3}, w_{t+2}, w_{t+1}, z_t} F_{z_{t+3}, \bar{w}_{t+2}, w_{t+1}, z_t}^{-1} F_{z_{t+3}, \bar{w}_{t+2}, \bar{w}_{t+1}, z_t} F_{z_{t+3}, w_{t+2}, \bar{w}_{t+1}, z_t}^{-1} \\ &= A_{z_{t+3}|w_{t+2}, \tau_{t+2}} D_{w_{t+2}, \bar{w}_{t+2}, w_{t+1}, \bar{w}_{t+1}|\tau_{t+2}} A_{z_{t+3}|w_{t+2}, \tau_{t+2}}^{-1}, \end{aligned} \quad (7)$$

where $D(w_{t+2}, \bar{w}_{t+2}, w_{t+1}, \bar{w}_{t+1}|\tau_{t+2}) = \frac{\Pr(w_{t+2}|w_{t+1}, \tau_{t+2}) \Pr(\bar{w}_{t+2}|\bar{w}_{t+1}, \tau_{t+2})}{\Pr(\bar{w}_{t+2}|w_{t+1}, \tau_{t+2}) \Pr(w_{t+2}|\bar{w}_{t+1}, \tau_{t+2})}$.

Equation (7) implies that the matrix $D_{w_{t+2}, \bar{w}_{t+2}, w_{t+1}, \bar{w}_{t+1}|\tau_{t+2}}$ is similar to the matrix in the left-hand side of the equation, which can be computed directly from the data. The eigenvector matrix $A_{z_{t+3}|w_{t+2}, \tau_{t+2}}$ is identified up to permutations of its columns. The following conditions are sufficient to

guarantee a unique decomposition.

Assumption 5. For $\{w_{t+2}, \bar{w}_{t+2}\}$, there exists a pair of $\{w_{t+1}, \bar{w}_{t+1}\}$, such that

- (1). $\Pr(\bar{w}_{t+2}|w_{t+1}, \tau_{t+2}) \Pr(w_{t+2}|\bar{w}_{t+1}, \tau_{t+2}) > 0$ for all τ_{t+2} ;
- (2). $D(w_{t+2}, \bar{w}_{t+2}, w_{t+1}, \bar{w}_{t+1}|\tau_{t+2} = i) \neq D(w_{t+2}, \bar{w}_{t+2}, w_{t+1}, \bar{w}_{t+1}|\tau_{t+2} = j)$ for any τ_{t+2} , and any $i \neq j$.

Condition (1) indicates that for any state combination in period $t+2$, there exists a state combination in period $t+1$, such that the transition between these two states is possible for any types in period $t+2$. This condition is not restrictive since it only requires the existence of one state combination in period $t+1$ for any state combination in period $t+2$.

Condition (2) indicates that the eigenvalues from the representation linking the observed and unobserved matrices are distinctive for uniqueness purpose. This condition is empirically testable because the matrix for the eigen-decomposition can be computed from the data.

Proposition 1. (*Identification of $\Pr(w_{t+3}|w_{t+2}, \tau_{t+2})$*): Given that Assumptions 4-5, and the conditions in Lemma 2 are satisfied, the following claims hold with four periods of data.

1. (**Permutation**) For any w_{t+2} , matrix $A_{w_{t+3}|w_{t+2}, \tau_{t+2}}$ is identified up to permutations of its columns.
2. (**Order preservation**) The permutations of columns for matrix $A_{w_{t+3}|w_{t+2}, \tau_{t+2}}$ can be preserved for different values of w_{t+2} .

Proof See Appendix ■

From Claim 1, for any values of w_{t+2} , the decomposition leads to a matrix $A_{w_{t+3}|w_{t+2}, \tau_{t+2}} I$, where I is an elementary matrix generated by interchanging columns of the identify matrix. Note that we have to conduct the decomposition for every possible values of w_{t+2} , which may result in different permutations of the columns. That is, the I matrix varies with w_{t+2} .

Claim 2 explores the identification structure to keep the permutation matrix I to be the same for different values of $w_t + 2$. This claim is the major econometric contribution in this paper, which is

novel and of practice importance. In the measurement error literature, a monotonicity conditions is imposed to determine the order of the latent variable, and the eigenvector matrix can be uniquely identified. However, the conventional monotonicity assumption is no longer applicable in the game setting, because τ_{t+2} combines information from both market-level unobserved heterogeneity and multiplicity of equilibria. Moreover, The monotonicity assumption is not necessarily valid, even when the data is generated by the same equilibrium. It is impossible to identify the payoff primitives if the order of the aggregate latent variable differs associated with different values of observables. Thus, it is particularly important to preserve the order, with which the payoff primitives can be identified up to the same order permutation. Pinning down the exact order of the aggregate latent variable is secondarily important.

In what follows, I show that all relevant components can be identified up to the same order permutation.

Lemma 3. (*Markov Law of Motion*) *Given that Assumptions 4-5, and the conditions in Lemma 2 are satisfied, for any combination of $\{w_{t+3}, w_{t+2}\}$, the Markov law of motion $A_{w_{t+3}, \tau_{t+3} | w_{t+2}, \tau_{t+2}}$ can be identified up to permutations of τ_{t+3} and τ_{t+2} with four periods of data.*

Proof See Appendix ■

Initial condition $\Pr(w_t, \tau_t)$ plays an important role in simulating the game to do estimation while this information is impossible to obtain from the data. However, following lemma states that the initial condition can also be uniquely recovered as a byproduct from the main identification.

Lemma 4. (*Initial Condition*): *Given that Assumptions 4-5, and the conditions in lemma 3 are satisfied, the initial density distribution $\Pr(w_t, \tau_t)$ can be identified up to a permutation of τ_t from four periods of data.*

Proof See Appendix ■

A byproduct of the identification of the initial distribution $\Pr(w_t, \tau_t)$ is the identification of the marginal distribution of the unobservables. The equilibrium selection, therefore, is identified once the

unobserved types and multiple equilibria are identified.

Lemma 5. *Given that Assumptions 4-5, and the conditions in Lemma 2 are satisfied, the policy function $\Pr(a_t|x_t, a_{t-1}, \tau_t)$, and the state transition for observables $\Pr(x_t|\tau_t, x_{t-1}, a_{t-1})$ and unobservables $\Pr(\tau_t|\tau_{t-1}, x_{t-1}, a_{t-1})$ can be identified up to permutations with four periods of data.*

Proof See Appendix ■

With CCPs and the state transition identified, payoff primitives can be identified following lemma 2 without multiple equilibria and unobserved heterogeneity using Assumptions 1-5. Identification proceeds first by identifying the differences in choice-specific per-period utility. Then with the exclusion restriction, the payoff primitives associated with each value of the latent variable $\pi_i(a_i, s, \tau)$ can be non-parametrically identified. The remaining issue is then how to distinguish between multiple equilibria and unobserved heterogeneity.

3.3 Second Step Identification - Payoff Primitives

In the first step identification, $\{\Pr(a_i|x, \tau), g(x_t, \tau_t|x_{t-1}, \tau_{t-1}, a_{t-1})\}$ are identified for all possible values of τ_t . Below I show that the payoff primitives associated with every possible value of τ ($\pi(a_i, a_{-i}, s, \tau)$) can be non-parametrically identified with exclusion restrictions, as in Bajari et al. (2009).

To non-parametrically identify the payoff functions, the following two assumptions are necessary, which are usually imposed in the existing literature.

Assumption 6. (*Normalization*) *For all i , all a_{-i} , all τ and all s , $\pi_i(a_i = 0, a_{-i}, s, \tau) = 0$.*

This assumption sets the mean utility from a particular choice as equal to zero, which is similar to the outside good condition in the discrete choice model. This assumption is a conventional assumption imposed in the existing literature.

Assumption 7. (*Exclusion Restriction*) *For each player i , the state variable can be partitioned into two parts denoted as $s = \{s_i, s_{-i}\}$, so that only s_i enters player i 's payoff, i.e. $\pi_i(a_i = k, a_{-i}, s, \tau) \equiv$*

$$\pi_i(a_i = k, a_{-i}, s_i, \tau).$$

An example of exclusion restrictions is a covariate, which shifts the profitability of one firm but can be excluded from the profits of all other firms. For example, firm-specific cost shifters are commonly used in empirical work. For example, Jia (2008) and Holmes (2011) demonstrate that distance from firm headquarters or distribution centers is a cost shifter for big box retailers such as Walmart.

Lemma 6. (*Identification of Payoff Primitives*) *Given that Assumptions 1-7, and the conditions in lemma 2 are satisfied, payoff primitives can be non-parametrically identified up to a permutation of the latent variable with four periods of data.*

Proof See Appendix ■

Note that the policy functions and state transitions are identified for each possible value of τ without knowing the true order of τ . Moreover, without extra information, we cannot distinguish between the unobserved market types or multiple equilibria through the identified policy functions or station transitions. However, since the order of the aggregate latent variable is preserved for different observables, the payoff primitives can be non-parametrically identified given any particular orders.

3.4 Distinguishing between Multiple Equilibrium and Unobserved Heterogeneity

The model is not fully identified without distinguishing between multiple equilibria and the unobserved heterogeneity. Comparing payoffs associated with two different values of τ helps distinguish market-level unobserved heterogeneity from multiple equilibria. Specifically, if the payoffs are the same for different values of τ , the two τ 's represent two different equilibria associated with the same market-type. If the payoffs are different, the two τ 's represent different market-types.

I denote the payoff function associated with type τ as $\pi_i(a_i, a_{-i}, s; \tau)$ where $\tau \in \{1, \dots, Q^\tau\}$. To distinguish between multiple equilibria and unobserved heterogeneity, the null hypothesis is specified as

$$H_0 : \pi_i(a_i, a_{-i}, s; \tau = l) = \pi_i(a_i, a_{-i}, s; \tau = k) \quad \forall a_i, i, s,$$

against the alternative

$$H_1 : \pi_i(a_i, a_{-i}, s; \tau = l) \neq \pi_i(a_i, a_{-i}, s; \tau = k) \quad \text{for one } a_i, i, x.$$

If we fail to reject the null hypothesis H_0 , $\tau = l$ and $\tau = k$ are two equilibria associated with the same market-type. Otherwise, $\tau = l$ and $\tau = k$ represent two different market types. Resting on the test results, I divide the Q^τ sets of payoff primitives into L groups with each group representing one unobserved market type.

As a byproduct of the comparison of payoffs, the cardinality of the unobserved heterogeneity can be identified as the number of groups. Additionally, the number of equilibria can be identified as the number of components in each group. Moreover, the equilibrium selection mechanism can be identified through the marginal distribution of τ , which is a conditional distribution within each group. In another word, all of the aspects of the game can be identified non-parametrically.

Theorem 2. (*Identification of Dynamic Games with Incomplete Information*) *Given that Assumptions 1-7, and the conditions in lemma 2 are satisfied, the cardinality and initial marginal distribution of the unobserved heterogeneity, the number of equilibria, the equilibrium selection, each player's equilibrium CCPs, state transitions of observables and unobservables, and payoff primitives are non-parametrically identified in dynamic games with four periods of data.*

3.5 Estimation

This section provides a brief discussion of the estimation which proceeds in sequential order with first estimating the cardinality of the overall latent variable and then the payoff primitives.

Estimation of the cardinality The cardinality of the latent factor τ can be estimated via estimating the rank of the matrix constructed by the joint distribution directly computable from the data, i.e. $R = \text{rank}(F_{w_{t+2}, \bar{w}_{t+1}, w_t})$, where the ij_{th} element of the matrix can be estimated using simple frequency

approach such as:

$$\Pr(w_{t+2} = w^i, \bar{w}_{t+1}, w_t = w^j) = \frac{I(w_{t+2} = w^i, \bar{w}_{t+1}, w_t = w^j)}{N}$$

where $I(\cdot)$ is the indicator function.

The cardinality of latent variable is estimated through estimating the rank of the joint distribution matrix $F_{w_{t+2}, \bar{w}_{t+1}, w_t}$, following the method developed in Robin and Smith (2000), which provides a test statistic using information on the characteristic roots of the matrix quadratic form.⁸ The estimation proceeds as a sequence of tests with a null hypothesis, in which the ranking of the unobserved matrix equates to a predetermined order. Specifically, the sequence of hypotheses are constructed as: $H_0^r : \text{rank}(F_{w_{t+2}, \bar{w}_{t+1}, w_t}) = r$ against the alternatives $H_1^r : \text{rank}(F_{w_{t+2}, \bar{w}_{t+1}, w_t}) > r$ with r starting at $r = 1$ and increasing. Since w_{t+2} contains all information from players' actions and observed state variable, the condition that dimension of matrix $F_{w_{t+2}, \bar{w}_{t+1}, w_t} > R$ should be easily satisfied.

Estimation of the structural parameters There are a few approaches developed for estimating dynamic games with unobserved heterogeneity. First, a plug-in estimator following the identification procedure in this paper is a nature alternative. Specifically, eigenvalue-eigenvector decomposition and simple algebra manipulation leads to estimators of the policy functions and state transitions, based on which the payoff primitive can be estimated via the least square estimator in Pesendorfer and Schmidt-Dengler (2008) or the simple simulated minimum distance estimator in Bajari et al. (2007). In addition, the payoff primitives and the policy functions and state transitions can be estimated together using the expectation-maximization algorithm in Arcidiacono and Miller (2011). For details of estimation, I refer to the existing literature.

4 Conclusion

This paper presents a methodology for non-parametrically identifying finite action games with incomplete information allowing for the presence of multiple equilibria and unobserved heterogeneity. Specifi-

⁸See also Kleibergen and Paap (2006) and Camba-Mendez and Kapetanios (2009) for a review.

cally, the cardinality of the overall latent factors can be identified non-parametrically. The law of motion and the equilibrium CCPs which are latent factor variant can also be uniquely recovered. Once the CCPs and transition functions have been identified, the payoffs can be non-parametrically identified with exclusion restrictions. Disentangling equilibria and unobserved heterogeneity can be obtained by testing the hypothesis that payoffs from different levels of latent variables are the same or different

Appendix

Following are proofs of lemmas and propositions presented in the paper.

A Proofs

Proof of Lemma 1 I first prove (i): $\{w_t, \eta_t\}$ follows a stationary first-order Markov chain. The distribution of state variables in period t conditioning on all of the history can be written as

$$\begin{aligned}
& \Pr(w_t, \eta_t | w_{t-1}, \eta_{t-1}, \Omega_{<t-1}) \\
= & \Pr(a_t, x_t, \eta_t | a_{t-1}, x_{t-1}, \eta_{t-1}, \Omega_{<t-1}) \\
= & \Pr(a_t | x_t, \eta_t, a_{t-1}, \Omega_{<t-1}) \Pr(x_t | \eta_t, x_{t-1}, \eta_{t-1}, a_{t-1}, \Omega_{<t-1}) \Pr(\eta_t | x_{t-1}, \eta_{t-1}, a_{t-1}, \Omega_{<t-1}) \\
= & \Pr(a_t | x_t, \eta_t, a_{t-1}) \Pr(x_t | \eta_t, x_{t-1}, a_{t-1}) \Pr(\eta_t | x_{t-1}, \eta_{t-1}, a_{t-1}) \\
= & \Pr(a_t, x_t, \eta_t | x_{t-1}, \eta_{t-1}, a_{t-1}) \\
= & \Pr(w_t, \eta_t | w_{t-1}, \eta_{t-1}).
\end{aligned}$$

The third equality holds because of assumption 1 and the Markov perfect equilibrium assumption. Moreover, given assumption 1, I can prove that for the observable, the history information of the unobserved state variable does not provide extra information when conditions on the current period's

information, i.e. $\Pr(w_{t+2}|\eta_{t+2}, w_{t+1}, \eta_{t+1}) = \Pr(w_{t+2}|\eta_{t+2}, w_{t+1})$.

$$\begin{aligned}
\Pr(w_{t+2}|\eta_{t+2}, w_{t+1}, \eta_{t+1}) &= \Pr(a_{t+2}, x_{t+2}|\eta_{t+2}, a_{t+1}, x_{t+1}, \eta_{t+1}) \\
&= \Pr(a_{t+2}|x_{t+2}, \eta_{t+2}, a_{t+1}, x_{t+1}, \eta_{t+1}) \Pr(x_{t+2}|\eta_{t+2}, a_{t+1}, x_{t+1}, \eta_{t+1}) \\
&= \Pr(a_{t+2}|x_{t+2}, \eta_{t+2}, a_{t+1}) \Pr(x_{t+2}|\eta_{t+2}, a_{t+1}, x_{t+1}) \\
&= \Pr(a_{t+2}, x_{t+2}|\eta_{t+2}, a_{t+1}, x_{t+1}) \\
&= \Pr(w_{t+2}|\eta_{t+2}, w_{t+1}).
\end{aligned} \tag{A.1}$$

With this Markovian property, I can express as follows the joint distribution of observables in three periods for a given market:

$$\begin{aligned}
\Pr(w_{t+2}, w_{t+1}, w_t) &= \sum_{\eta_{t+1}} \Pr(w_{t+2}, w_{t+1}, \eta_{t+1}, w_t) \\
&= \sum_{\eta_{t+1}} \Pr(w_{t+2}|w_{t+1}, \eta_{t+1}) \Pr(w_{t+1}, w_t|\eta_{t+1}) \Pr(\eta_{t+1}).
\end{aligned} \tag{A.2}$$

In a given market, the joint distribution of four periods of observables can be represented as follows

$$\begin{aligned}
&\Pr(w_{t+3}, w_{t+2}, w_{t+1}, w_t) \\
&= \sum_{\eta_{t+2}, \eta_{t+1}} \Pr(w_{t+3}, w_{t+2}, \eta_{t+2}, w_{t+1}, \eta_{t+1}, w_t) \\
&= \sum_{\eta_{t+2}, \eta_{t+1}} \Pr(w_{t+3}|w_{t+2}, \eta_{t+2}) \Pr(w_{t+2}, \eta_{t+2}|w_{t+1}, \eta_{t+1}) \Pr(w_{t+1}, \eta_{t+1}, w_t) \\
&= \sum_{\eta_{t+2}, \eta_{t+1}} \Pr(w_{t+3}|w_{t+2}, \eta_{t+2}) \Pr(w_{t+2}|\eta_{t+2}, w_{t+1}, \eta_{t+1}) \Pr(\eta_{t+2}|w_{t+1}, \eta_{t+1}) \Pr(w_{t+1}, \eta_{t+1}, w_t) \\
&= \sum_{\eta_{t+2}, \eta_{t+1}} \Pr(w_{t+3}|w_{t+2}, \eta_{t+2}) \Pr(w_{t+2}|\eta_{t+2}, w_{t+1}) \Pr(\eta_{t+2}|w_{t+1}, \eta_{t+1}) \Pr(w_{t+1}, \eta_{t+1}, w_t) \\
&= \sum_{\eta_{t+2}, \eta_{t+1}} \Pr(w_{t+3}|w_{t+2}, \eta_{t+2}) \Pr(w_{t+2}|\eta_{t+2}, w_{t+1}) \Pr(\eta_{t+2}, w_{t+1}, \eta_{t+1}, w_t) \\
&= \sum_{\eta_{t+2}} \Pr(w_{t+3}|w_{t+2}, \eta_{t+2}) \Pr(w_{t+2}|\eta_{t+2}, w_{t+1}) \sum_{\eta_{t+1}} \Pr(\eta_{t+2}, w_{t+1}, \eta_{t+1}, w_t) \\
&= \sum_{\eta_{t+2}} \Pr(w_{t+3}|w_{t+2}, \eta_{t+2}) \Pr(w_{t+2}|\eta_{t+2}, w_{t+1}) \Pr(w_{t+1}, w_t, \eta_{t+2}).
\end{aligned} \tag{A.3}$$

■

Proof of Lemma 2 Based on the MPE assumption and assumption 1, we have the following joint distribution

$$F_{w_{t+2}, \bar{w}_{t+1}, w_t} = A_{w_{t+2}|\bar{w}_{t+1}, \tau_{t+1}} D_{\tau_{t+1}} A_{\bar{w}_{t+1}, w_t|\tau_{t+1}}. \quad (\text{A.4})$$

Given the assumptions that $\|X\| \times \|A^n\| > Q^\tau$ and full rank of both matrices $A_{w_{t+2}|\bar{w}_{t+1}, \tau_{t+1}}$ and $A_{\bar{w}_{t+1}, w_t|\tau_{t+1}}$, then, according to the following inequality regarding the rank of matrix $F_{w_{t+2}, \bar{w}_{t+1}, w_t}$

$$\begin{aligned} & \text{Rank}(A_{w_{t+2}|\bar{w}_{t+1}, \tau_{t+1}}) + \text{Rank}(A_{\bar{w}_{t+1}, w_t|\tau_{t+1}}) - Q^\tau \\ & \leq F_{w_{t+2}, \bar{w}_{t+1}, w_t} \leq \min\{\text{Rank}(A_{w_{t+2}|\bar{w}_{t+1}, \tau_{t+1}}), \text{Rank}(A_{\bar{w}_{t+1}, w_t|\tau_{t+1}})\}. \end{aligned}$$

I conclude that $\text{Rank}(F_{w_{t+2}, \bar{w}_{t+1}, w_t}) = Q^\tau$.

■

Proof of Proposition 1 I first show that for any value of w_{t+2} , $\Pr(z_{t+3}|w_{t+2}, \tau_{t+2})$ can be identified up to ordering. Fixing w_{t+2} and w_{t+1} , matrix $F_{z_{t+3}, w_{t+2}, w_{t+1}, z_t}$ defined as below, is invertible. As a result, matrices $A_{w_{t+3}|w_{t+2}, \tau_{t+2}}$ and $B_{w_{t+1}, w_t, \tau_{t+2}}$ are also invertible. Evaluating the joint distribution of four periods of data at the four pairs of points (w_{t+2}, w_{t+1}) , $(\bar{w}_{t+2}, w_{t+1})$, $(w_{t+2}, \bar{w}_{t+1})$, $(\bar{w}_{t+2}, \bar{w}_{t+1})$, each pair of equations will share one matrix in common. Specifically,

$$(w_{t+2}, w_{t+1}) : F_{z_{t+3}, w_{t+2}, w_{t+1}, z_t} = A_{z_{t+3}|w_{t+2}, \tau_{t+2}} D_{w_{t+2}|w_{t+1}, \tau_{t+2}} B_{w_{t+1}, z_t, \tau_{t+2}} \quad (\text{A.5})$$

$$(\bar{w}_{t+2}, w_{t+1}) : F_{z_{t+3}, \bar{w}_{t+2}, w_{t+1}, z_t} = A_{z_{t+3}|\bar{w}_{t+2}, \tau_{t+2}} D_{\bar{w}_{t+2}|w_{t+1}, \tau_{t+2}} B_{w_{t+1}, z_t, \tau_{t+2}} \quad (\text{A.6})$$

$$(w_{t+2}, \bar{w}_{t+1}) : F_{z_{t+3}, w_{t+2}, \bar{w}_{t+1}, z_t} = A_{z_{t+3}|w_{t+2}, \tau_{t+2}} D_{w_{t+2}|\bar{w}_{t+1}, \tau_{t+2}} B_{\bar{w}_{t+1}, z_t, \tau_{t+2}} \quad (\text{A.7})$$

$$(\bar{w}_{t+2}, \bar{w}_{t+1}) : F_{z_{t+3}, \bar{w}_{t+2}, \bar{w}_{t+1}, z_t} = A_{z_{t+3}|\bar{w}_{t+2}, \tau_{t+2}} D_{\bar{w}_{t+2}|\bar{w}_{t+1}, \tau_{t+2}} B_{\bar{w}_{t+1}, z_t, \tau_{t+2}} \quad (\text{A.8})$$

Matrices $A_{z_{t+3}|w_{t+2}, \tau_{t+2}}$ and $B_{w_{t+1}, z_t, \tau_{t+2}}$ are invertible by construction. Assume that $\Pr(w_{t+2}|w_{t+1}, \tau_{t+2})$ is positive for every combination of w_{t+2} and w_{t+1} ; then matrix $D_{w_{t+2}|w_{t+1}, \tau_{t+2}}$ is also invertible. Con-

sequently, we can post-multiply the inverse of equation A.6 to equation A.5, to obtain

$$Y \equiv F_{z_{t+3}, w_{t+2}, w_{t+1}, z_t} F_{z_{t+3}, \bar{w}_{t+2}, w_{t+1}, z_t}^{-1} = A_{z_{t+3}|w_{t+2}, \tau_{t+2}} D_{w_{t+2}|w_{t+1}, \tau_{t+2}} D_{\bar{w}_{t+2}|w_{t+1}, \tau_{t+2}}^{-1} A_{z_{t+3}|\bar{w}_{t+2}, \tau_{t+2}}^{-1}. \quad (\text{A.9})$$

Similarly,

$$Z \equiv F_{z_{t+3}, \bar{w}_{t+2}, \bar{w}_{t+1}, z_t} F_{z_{t+3}, w_{t+2}, \bar{w}_{t+1}, z_t}^{-1} = A_{z_{t+3}|\bar{w}_{t+2}, \tau_{t+2}} D_{\bar{w}_{t+2}|\bar{w}_{t+1}, \tau_{t+2}} D_{w_{t+2}|\bar{w}_{t+1}, \tau_{t+2}}^{-1} A_{z_{t+3}|w_{t+2}, \tau_{t+2}}^{-1}. \quad (\text{A.10})$$

Consequently, I postmultiply Eq. A.9 by Eq. A.10, leading to

$$\begin{aligned} YZ &= A_{z_{t+3}|w_{t+2}, \tau_{t+2}} \left(D_{w_{t+2}|w_{t+1}, \tau_{t+2}} D_{\bar{w}_{t+2}|w_{t+1}, \tau_{t+2}}^{-1} D_{\bar{w}_{t+2}|\bar{w}_{t+1}, \tau_{t+2}} D_{w_{t+2}|\bar{w}_{t+1}, \tau_{t+2}}^{-1} \right) A_{z_{t+3}|w_{t+2}, \tau_{t+2}}^{-1} \\ &\equiv A_{z_{t+3}|w_{t+2}, \tau_{t+2}} D_{w_{t+2}, \bar{w}_{t+2}, w_{t+1}, \bar{w}_{t+1}|\tau_{t+2}} A_{z_{t+3}|w_{t+2}, \tau_{t+2}}^{-1}, \quad \text{where} \end{aligned} \quad (\text{A.11})$$

$$\begin{aligned} D_{w_{t+2}, \bar{w}_{t+2}, w_{t+1}, \bar{w}_{t+1}|\tau_{t+2}} &\equiv D_{w_{t+2}|w_{t+1}, \tau_{t+2}} D_{\bar{w}_{t+2}|w_{t+1}, \tau_{t+2}}^{-1} D_{\bar{w}_{t+2}|\bar{w}_{t+1}, \tau_{t+2}} D_{w_{t+2}|\bar{w}_{t+1}, \tau_{t+2}}^{-1} \\ &= \frac{Pr(w_{t+2}|w_{t+1}, \tau_{t+2}) Pr(\bar{w}_{t+2}|\bar{w}_{t+1}, \tau_{t+2})}{Pr(\bar{w}_{t+2}|w_{t+1}, \tau_{t+2}) Pr(w_{t+2}|\bar{w}_{t+1}, \tau_{t+2})}. \end{aligned}$$

The matrix on the left-hand side of equation A.11 can be directly computed from the data, while the matrices on the right-hand side are of particular interest. Moreover, this representation indicates that the joint distribution of observables on the left-hand side of the equation includes the same eigenvalue-eigenvector decomposition of the unknown matrix on the right-hand side. Consequently, $D_{w_{t+2}, \bar{w}_{t+2}, w_{t+1}, \bar{w}_{t+1}|\tau_{t+2}}$ can be identified as eigenvalues up to permutations of columns, and $A_{w_{t+3}|w_{t+2}, \tau_{t+2}}$ can be identified as eigenvectors up to scale and permutations of columns. Since each column in the matrix $A_{z_{t+3}|w_{t+2}, \tau_{t+2}}$ represents an entire distribution, the column sum should equal to 1, basing on which normalization can be performed. The decomposition leads to a matrix $A_{w_{t+3}|w_{t+2}, \tau_{t+2}} I$, where I is an elementary matrix generated by interchanging columns of the identify matrix. Note that we have to conduct the decomposition for every possible values of w_{t+2} , which may result in different permutations of the columns. That is, the I matrix varies with w_{t+2} .

The goal is to identify $Pr(w_{t+3}|w_{t+2}, \tau_{t+2})$ instead of $Pr(z_{t+3}|w_{t+2}, \tau_{t+2})$, which is an aggregation of

the former counterpart. $\Pr(w_{t+3}|w_{t+2}, \tau_{t+2})$ is also identified up to permutation for every w_{t+2} . For every value of w_{t+2} , equation (A.4) leads to

$$\begin{aligned} F_{z_{t+3}, w_{t+2}, z_{t+1}} &= A_{z_{t+3}|w_{t+2}, \tau_{t+2}} A_{w_{t+2}, \tau_{t+2}, z_{t+1}}, \\ F_{w_{t+3}, w_{t+2}, z_{t+1}} &= A_{w_{t+3}|w_{t+2}, \tau_{t+2}} A_{w_{t+2}, \tau_{t+2}, z_{t+1}}. \end{aligned}$$

Thus, matrix $A_{w_{t+3}|w_{t+2}, \tau_{t+2}}$ can be identified up to permutation of columns through the following equation

$$A_{w_{t+3}|w_{t+2}, \tau_{t+2}} I = F_{w_{t+3}, w_{t+2}, z_{t+1}} F_{z_{t+3}, w_{t+2}, z_{t+1}}^{-1} A_{z_{t+3}|w_{t+2}, \tau_{t+2}} I.$$

I now move to show that the permutation of columns can be preserved for different values of w_{t+2} . Matrix $A_{w_{t+3}|w_{t+2}, \tau_{t+2}}$ is identified through evaluating the joint distribution of four periods of data at four pairs of points (w_{t+2}, w_{t+1}) , $(\bar{w}_{t+2}, w_{t+1})$, $(w_{t+2}, \bar{w}_{t+1})$, $(\bar{w}_{t+2}, \bar{w}_{t+1})$ and as the eigenvectors of the matrix on the left-hand side based on Equation A.11.

Similarly, postmultiplying Eq. A.10 by Eq. A.9 leads to

$$\begin{aligned} ZY &= A_{z_{t+3}|\bar{w}_{t+2}, \tau_{t+2}} \left(D_{\bar{w}_{t+2}|\bar{w}_{t+1}, \tau_{t+2}} D_{w_{t+2}|\bar{w}_{t+1}, \tau_{t+2}}^{-1} D_{w_{t+2}|w_{t+1}, \tau_{t+2}} D_{\bar{w}_{t+2}|w_{t+1}, \tau_{t+2}}^{-1} \right) A_{w_{t+3}|\bar{w}_{t+2}, \tau_{t+2}}^{-1} \\ &\equiv A_{z_{t+3}|\bar{w}_{t+2}, \tau_{t+2}} D_{\bar{w}_{t+2}, w_{t+2}, \bar{w}_{t+1}, w_{t+1}|\tau_{t+2}} A_{w_{t+3}|\bar{w}_{t+2}, \tau_{t+2}}^{-1}, \quad \text{where} \end{aligned} \quad (\text{A.12})$$

$$D_{\bar{w}_{t+2}, w_{t+2}, \bar{w}_{t+1}, w_{t+1}|\tau_{t+2}} = \frac{\Pr(w_{t+2}|w_{t+1}, \tau_{t+2}) \Pr(\bar{w}_{t+2}|\bar{w}_{t+1}, \tau_{t+2})}{\Pr(\bar{w}_{t+2}|w_{t+1}, \tau_{t+2}) \Pr(w_{t+2}|\bar{w}_{t+1}, \tau_{t+2})}.$$

Importantly, the diagonal matrix for the right hand-side of equations A.11 and A are the same, which can be used to preserve permutation matrix I for values w_{t+2} and $\bar{w}_{t+2}$. Specifically, from Equations A.11 and , I can identify

$$(A_{w_{t+3}|w_{t+2}, \tau_{t+2}} I_{w_{t+2}}, I_{w_{t+2}} D_{w_{t+2}} I_{w_{t+2}}^{-1}),$$

and

$$(A_{w_{t+3}|\bar{w}_{t+2}, \tau_{t+2}} I_{\bar{w}_{t+2}}, I_{\bar{w}_{t+2}} D_{\bar{w}_{t+2}} I_{\bar{w}_{t+2}}^{-1}),$$

respectively, where the permutation might vary with w_{t+2} . Note that the eigenvalue matrices are the same by construction, i.e., $D_{w_{t+2}} = D_{\bar{w}_{t+2}}$. If we force the eigenvalue matrices from the decomposition to be the same ($I_{w_{t+2}} D_{w_{t+2}} I_{w_{t+2}}^{-1} = I_{\bar{w}_{t+2}} D_{\bar{w}_{t+2}} I_{\bar{w}_{t+2}}^{-1}$), the permutations in the two cases are the same ($I_{w_{t+2}} = I_{\bar{w}_{t+2}}$). Consequently, exploring the identification structure, the permutations of the latent variable can be preserved for different values of observables w_{t+2} .

To illustrate the intuition, assume that the latent factor $\tau \in \{H, L\}$ is time invariant. Specifically, to identify $\Pr(z_{t+3}|w_{t+2} = 0, \tau)$, we do eigen-decomposition with respect to the observed matrix YZ , leading to two possible results

$$\begin{aligned} YZ &= [\Pr(z_{t+3}|0, \tau = L) \ \Pr(z_{t+3}|0, \tau = H)] \begin{pmatrix} f(\tau = L) & 0 \\ 0 & f(\tau = H) \end{pmatrix} \begin{pmatrix} \Pr(z_{t+3}|0, \tau = L)^T \\ \Pr(z_{t+3}|0, \tau = H)^T \end{pmatrix}, \\ YZ &= [\Pr(z_{t+3}|0, \tau = H) \ \Pr(z_{t+3}|0, \tau = L)] \begin{pmatrix} f(\tau = H) & 0 \\ 0 & f(\tau = L) \end{pmatrix} \begin{pmatrix} \Pr(z_{t+3}|0, \tau = H)^T \\ \Pr(z_{t+3}|0, \tau = L)^T \end{pmatrix}. \end{aligned}$$

Similarly, decomposition with respect to the observed matrix ZY leads to

$$\begin{aligned} ZY &= [\Pr(z_{t+3}|1, \tau = L) \ \Pr(z_{t+3}|1, \tau = H)] \begin{pmatrix} f(\tau = L) & 0 \\ 0 & f(\tau = H) \end{pmatrix} \begin{pmatrix} \Pr(z_{t+3}|1, \tau = L)^T \\ \Pr(z_{t+3}|1, \tau = H)^T \end{pmatrix}, \\ ZY &= [\Pr(z_{t+3}|1, \tau = H) \ \Pr(z_{t+3}|1, \tau = L)] \begin{pmatrix} f(\tau = H) & 0 \\ 0 & f(\tau = L) \end{pmatrix} \begin{pmatrix} \Pr(z_{t+3}|1, \tau = H)^T \\ \Pr(z_{t+3}|1, \tau = L)^T \end{pmatrix}. \end{aligned}$$

Without information from the eigenvalue matrix, there are four possible matches after the decomposition such as $\{[L \ H][L \ H]\}$, $\{[L \ H][H \ L]\}$, $\{[H \ L][LH]\}$, and $\{[H \ L][H \ L]\}$, among which matches $\{[L \ H][L \ H]\}$ and $\{[H \ L][H \ L]\}$ are consistent, while $\{[L \ H][H \ L]\}$ and $\{[H \ L][LH]\}$ are inconsistent. However, the diagonal matrices from the two compositions should be the same when there are consistent matches in both cases. As a result, we can rule out the inconsistent matches case of $\{[L \ H][H \ L]\}$,

$\{[H \ L][LH]\}$. Yet, without further assumptions, we are not able to recover the order of the unobserved latent factor; thus, , we still have the relabeling issue, but it does not affect the identification of the payoff primitives.

Furthermore, for other values of $w_{t+2} = \underline{w}_{t+2}$, one can uses a similar logic in exploring the four pairs of (w_{t+2}, w_{t+1}) , $(\underline{w}_{t+2}, w_{t+1})$, $(w_{t+2}, \bar{w}_{t+1})$, $(\underline{w}_{t+2}, \bar{w}_{t+1})$. ■

Proof of Lemma 3: (Identification of Law of Motion) Again, with four periods of data, the joint distribution of observables can be factorized as the components that we want to identify in the following equations:

$$\Pr(z_{t+3}, w_{t+2}, w_{t+1}, z_t) = \sum_{\tau_{t+2}} \Pr(z_{t+3}|w_{t+2}, \tau_{t+2}) \Pr(w_{t+2}, \tau_{t+2}, w_{t+1}, z_t), \quad (\text{A.13})$$

$$\begin{aligned} \Pr(w_{t+2}, \tau_{t+2}, w_{t+1}, z_t) &= \sum_{\tau_{t+1}} \Pr(w_{t+2}, \tau_{t+2}, w_{t+1}, \tau_{t+1}, z_t) \\ &= \sum_{\tau_{t+1}} \Pr(w_{t+2}, \tau_{t+2}|w_{t+1}, \tau_{t+1}) \Pr(w_{t+1}, \tau_{t+1}, z_t). \end{aligned} \quad (\text{A.14})$$

Fixing $w_{t+2} = \bar{w}_{t+2}$ and $w_{t+1} = \bar{w}_{t+1}$, and rewrite above equations into a matrix format similar to that defined at the outset:

$$F_{z_{t+3}, \bar{w}_{t+2}, \bar{w}_{t+1}, z_t} = A_{z_{t+3}|\bar{w}_{t+2}, \tau_{t+2}} B_{\bar{w}_{t+2}, \tau_{t+2}, \bar{w}_{t+1}, z_t}, \quad (\text{A.15})$$

$$B_{\bar{w}_{t+2}, \tau_{t+2}, \bar{w}_{t+1}, z_t} = A_{\bar{w}_{t+2}, \tau_{t+2}|\bar{w}_{t+1}, \tau_{t+1}} A_{\bar{w}_{t+1}, \tau_{t+1}, z_t}. \quad (\text{A.16})$$

Consequently, we have

$$F_{z_{t+3}, \bar{w}_{t+2}, \bar{w}_{t+1}, z_t} = A_{z_{t+3}|\bar{w}_{t+2}, \tau_{t+2}} A_{\bar{w}_{t+2}, \tau_{t+2}|\bar{w}_{t+1}, \tau_{t+1}} A_{\bar{w}_{t+1}, \tau_{t+1}, z_t}. \quad (\text{A.17})$$

I show in lemma 1 that $A_{z_{t+3}|\bar{w}_{t+2}, \tau_{t+2}}$ is identified up to a permutation of its columns; that is, we can only know $A_{z_{t+3}|\bar{w}_{t+2}, \tau_{t+2}} I$, where I is an elementary matrix generated by interchanging columns of the identify matrix. Moreover, based on Lemma 2, the permutation matrix I is invariant of W_{t+2} . Additionally, the left-land side matrix can be computed from the data.

Identification of the law of motion depends on the identification of matrix $A_{\bar{w}_{t+1}, \tau_{t+1}, z_t}$, because both $A_{\bar{w}_{t+1}, \tau_{t+1}, z_t}$ and $A_{z_{t+3}|\bar{w}_{t+2}, \tau_{t+2}}$ are invertible by construction. I next show that $A_{\bar{w}_{t+1}, \tau_{t+1}, z_t}$ can be identified again up to a permutation through the following equation:

$$\Pr(z_{t+2}, w_{t+1}, z_t) = \sum_{\tau_{t+1}} \Pr(z_{t+2}|w_{t+1}, \tau_{t+1}) \Pr(w_{t+1}, \tau_{t+1}, z_t). \quad (\text{A.18})$$

Using a similar logic, fixing $w_{t+1} = \bar{w}_{t+1}$, the above equation's matrix counterpart is as follows:

$$F_{z_{t+2}, \bar{w}_{t+1}, z_t} = A_{z_{t+2}|\bar{w}_{t+1}, \tau_{t+1}} A_{\bar{w}_{t+1}, \tau_{t+1}, z_t}. \quad (\text{A.19})$$

$A_{w_{t+2}|\bar{w}_{t+1}, \tau_{t+1}}$ is identified up to a permutation of I by the stationary assumption and is invertible. That is, we can identify $A_{w_{t+2}|\bar{w}_{t+1}, \tau_{t+1}} I$. Consequently, matrix $A_{\bar{w}_{t+1}, \tau_{t+1}, z_t}$ is identified up to a permutation of rows. That is, $I^{-1} A_{\bar{w}_{t+1}, \tau_{t+1}, z_t}$ is identified. Substituting the identified matrix back to equation ??, we have

$$F_{z_{t+3}, \bar{w}_{t+2}, \bar{w}_{t+1}, z_t} = A_{z_{t+3}|\bar{w}_{t+2}, \tau_{t+2}} I I^{-1} A_{\bar{w}_{t+2}, \tau_{t+2}|\bar{w}_{t+1}, \tau_{t+1}} I I^{-1} A_{\bar{w}_{t+1}, \tau_{t+1}, z_t}. \quad (\text{A.20})$$

Consequently, the law of motion is identified up to a permutation of both rows and columns:

$$I^{-1} A_{\bar{w}_{t+2}, \tau_{t+2}|\bar{w}_{t+1}, \tau_{t+1}} I.$$

■

Proof of Lemma 4: (Identification of the Initial Condition) Given that we have already identified the transition matrix $\Pr(w_{t+1}|w_t, \tau_t)$, the following equation provides identification of the initial distribution

$$\Pr(z_{t+1}, w_t) = \sum_{\tau_t} \Pr(z_{t+1}|w_t, \tau_t) \Pr(w_t, \tau_t). \quad (\text{A.21})$$

Fix $w_t = \bar{w}_t$, and rewrite above equation in matrix format

$$V_{z_{t+1}, \bar{w}_t} = A_{z_{t+1}|\bar{w}_t, \tau_t} V_{\bar{w}_t, \tau_t}. \quad (\text{A.22})$$

Since $A_{z_{t+1}|\bar{w}_t, \tau_t}$ is invertible and identified up to a permutation of columns, $V_{\bar{w}_t, \tau_t}$ is identified up to a permutation of rows for all w_t . ■

Proof of Lemma 5: (Identification of the Policy Functions and Transition Functions) With the identification of the Law of Motion, we can easily identify the transition functions for unobservables τ_{t+1} through the following equations:

$$\Pr(\tau_{t+1}|x_t, \tau_t, a_t) = \sum_{x_{t+1}, a_{t+1}} \Pr(x_{t+1}, \tau_{t+1}, a_{t+1}|x_t, \tau_t, a_t) = \sum_{w_{t+1}} \Pr(w_{t+1}, \tau_{t+1}|w_t, \tau_t), \quad (\text{A.23})$$

Note that the policy function $\Pr(\tau_{t+1}|x_t, \tau_t, a_t)$ is identified up to a permutation of τ_{t+1} and τ_t .

The transition function for the observed state variable can be identified up to a permutation of τ_{t+1} through the following equation:

$$\Pr(x_{t+1}|\tau_{t+1}, x_t, a_t) = \frac{\Pr(x_{t+1}, \tau_{t+1}|x_t, a_t, \tau_t)}{\Pr(\tau_{t+1}|x_t, \tau_t, a_t)} = \frac{\Pr(w_{t+1}|w_t, \tau_t)}{\Pr(\tau_{t+1}|x_t, \tau_t, a_t)}. \quad (\text{A.24})$$

The policy function $\Pr(a_{t+1}|x_{t+1}, \tau_{t+1}, a_t)$ can be identified up to a permutation of τ_{t+1} through the following equation:

$$\Pr(a_{t+1}|x_{t+1}, \tau_{t+1}, a_t) = \frac{\Pr(a_{t+1}, x_{t+1}, \tau_{t+1}|a_t, x_t, \tau_t)}{\Pr(x_{t+1}, \tau_{t+1}|a_t, x_t, \tau_t)} = \frac{\Pr(a_{t+1}, x_{t+1}, \tau_{t+1}|a_t, x_t, \tau_t)}{\sum_{a_{t+1}} \Pr(a_{t+1}, x_{t+1}, \tau_{t+1}|a_t, x_t, \tau_t)}. \quad (\text{A.25})$$
■

Proof of Lemma 6: (Identification of Payoff Primitives) The equilibrium condition with extreme value distribution assumption on the private shocks becomes

$$\log(p_i(a_i = k, s)) - \log(p_i(a_i = 0, s)) = V_i(a_i = k, s) - V_i(a_i = 0, s). \quad (\text{A.26})$$

The definition of the ex ante value function for player i leads to

$$\begin{aligned}
V_i(s) &= E_{\epsilon_i} \max_{a_i} (V_i(a_i, s) + \epsilon_i(a_i)) = \log\left(\sum_{k=0}^K \exp(V_i(k, s))\right) \\
&= \log \sum_{k=0}^K \exp(V_i(k, s) - V_i(0, s)) + V_i(0, s).
\end{aligned} \tag{A.27}$$

The second equation holds because ϵ_i is assumed to be i.i.d. extreme value distributed. By definition, we can relate the ex ante value function and choice-specific value functions through the following equation:

$$\begin{aligned}
V_i(a_i, s) &= \Pi_i(a_i, s) + \beta E_{s'}(V_i(s')|s, a_i) \\
&= \Pi_i(a_i, s) + \beta E_{s'}\left(\log \sum_{k=0}^K \exp(V_i(k, s') - V_i(0, s'))|s, a_i + V_i(0, s')|s, a_i\right) \\
&= \Pi_i(a_i, s) + \beta E_{s'}\left(\log \sum_{k=0}^K \exp(V_i(k, s') - V_i(0, s'))|s, a_i + \beta E_{s'}(V_i(0, s')|s, a_i)\right).
\end{aligned} \tag{A.28}$$

The normalization assumption of zero utility for option 0 yields

$$\begin{aligned}
V_i(0, s) &= \Pi_i(a_i = 0, s) + \beta E\left(\log \sum_{k=0}^K \exp(V_i(k, s) - V_i(0, s))|a_i, s + \beta E(V_i(0, s')|s, a_i = 0)\right) \\
&= \beta E_{s'}\left(\log \sum_{k=0}^K \exp(V_i(k, s') - V_i(0, s'))|a_i, s + \beta E_{s'}(V_i(0, s')|s, a_i = 0)\right).
\end{aligned} \tag{A.29}$$

Given the equilibrium condition, the term $(\log \sum_{k=0}^K \exp(V_i(k, s) - V_i(0, s)))$ can be computed through $\log(\sigma_i(a_i = k, s)) - \log(\sigma_i(a_i = 0, s))$, so we can treat it as a constant. Consequently, equation A.29 provides a contract mapping on unknowns $V_i(0, s)$. A unique fixed point is guaranteed based on Blackwell's condition. As a result, $V_i(0, S)$ is identified, and the ex ante value function $V_i(s)$ is identified. $V_i(a_i = k, s)$ is identified based on equation A.29. Consequently, we can identify the expected period payoff $\Pi_i(a_i, s)$ through equation A.28 as:

$$\Pi_i(a_i, s) = V_i(a_i, s) - \beta E_{s'}\left(\log \sum_{k=0}^K \exp(V_i(k, s) - V_i(0, s))|a_i, s - \beta E(V_i(0, s')|s, a_i)\right). \tag{A.30}$$

With the identification of the expected period payoff defined as $\Pi_i(a_i, s) = \sum_{a_{-i}} \pi_i(a_i, a_{-i}, s) \sigma_{-i}(a_{-i}|s)$, the identification of payoff functions $\pi_i(a_i, a_{-i}, s)$ is exactly the same as in the static case with exclusion restrictions. ■

References

- Aguirregabiria, Victor and Pedro Mira**, “Identification and Estimation of Dynamics Discrete Games,” in “manuscript presented at the Conference on Industrial Organization and the Food Processing Industry, Institut d’Economie Industrielle (IDEI). Toulouse, France” 2002.
- **and** —, “Sequential estimation of dynamic discrete games,” *Econometrica*, 2007, *75* (1), 1–53.
- **and** —, “Identification of Games of Incomplete Information with Multiple Equilibria and Common Unobserved Heterogeneity,” Technical Report 2015.
- Andrew, Alan L, K-W Eric Chu, and Peter Lancaster**, “Derivatives of eigenvalues and eigenvectors of matrix functions,” *SIAM journal on matrix analysis and applications*, 1993, *14* (4), 903–926.
- Arcidiacono, Peter and Robert A Miller**, “Conditional choice probability estimation of dynamic discrete choice models with unobserved heterogeneity,” *Econometrica*, 2011, *79* (6), 1823–1867.
- Bajari, P., H. Hong, and S.P. Ryan**, “Identification and estimation of a discrete game of complete information,” *Econometrica*, 2010, *78* (5), 1529–1568.
- Bajari, Patrick, C Lanier Benkard, and Jonathan Levin**, “Estimating dynamic models of imperfect competition,” *Econometrica*, 2007, *75* (5), 1331–1370.
- , **Victor Chernozhukov, Han Hong, and Denis Nekipelov**, “Nonparametric and Semiparametric Analysis of a Dynamic Discrete Game,” *Unpublished manuscript, Dep. Econ., Univ. Minn., Minneapolis*, 2009.

- Camba-Mendez, Gonzalo and George Kapetanios**, “Statistical tests and estimators of the rank of a matrix and their applications in econometric modelling,” *Econometric Reviews*, 2009, 28 (6), 581–611.
- Ciliberto, Federico and Elie Tamer**, “Market structure and multiple equilibria in airline markets,” *Econometrica*, 2009, 77 (6), 1791–1828.
- Egesdal, Michael, Zhenyu Lai, and Che-Lin Su**, “Estimating Dynamic Discrete-Choice Games of Incomplete Information,” *Quantitative Economics*, 2015.
- Ericson, Richard and Ariel Pakes**, “Markov-perfect industry dynamics: A framework for empirical work,” *The Review of Economic Studies*, 1995, 62 (1), 53–82.
- Haller, Hans and Roger Lagunoff**, “Genericity and Markovian behavior in stochastic games,” *Econometrica*, 2000, 68 (5), 1231–1248.
- Holmes, Thomas J**, “The Diffusion of Wal-Mart and Economies of Density,” *Econometrica*, 2011, 79 (1), 253–302.
- Hosoya, Yuzo**, “Hierarchical Statistical Models and a Generalized Likelihood Ratio Test,” *Journal of the Royal Statistical Society. Series B (Methodological)*, 1989, 51 (3), pp. 435–447.
- Hotz, V Joseph and Robert A Miller**, “Conditional choice probabilities and the estimation of dynamic models,” *The Review of Economic Studies*, 1993, 60 (3), 497–529.
- Hu, Yingyao and Matthew Shum**, “Nonparametric identification of dynamic models with unobserved state variables,” *Journal of Econometrics*, 2012, 171 (1), 32–44.
- Igami, Mitsuru and Nathan Yang**, “Unobserved Heterogeneity in Dynamic Games: Cannibalization and Preemptive Entry of Hamburger Chains in Canada,” *Quantitative Economics, Forthcoming*, 2015.
- Jia, P.**, “What Happens When Wal-Mart Comes to Town: An Empirical Analysis of the Discount Retailing Industry,” *Econometrica*, 2008, 76 (6), 1263–1316.

- Jovanovic, Boyan**, “Observable implications of models with multiple equilibria,” *Econometrica: Journal of the Econometric Society*, 1989, pp. 1431–1437.
- Kasahara, Hiroyuki and Katsumi Shimotsu**, “Nonparametric identification of finite mixture models of dynamic discrete choices,” *Econometrica*, 2009, *77* (1), 135–175.
- **and —**, “Non-parametric identification and estimation of the number of components in multivariate mixtures,” *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 2014, *76* (1), 97–111.
- Kleibergen, Frank and Richard Paap**, “Generalized reduced rank tests using the singular value decomposition,” *Journal of Econometrics*, 2006, *133* (1), 97–126.
- Milgrom, Paul R and Robert J Weber**, “Distributional strategies for games with incomplete information,” *Mathematics of Operations Research*, 1985, *10* (4), 619–632.
- Nevo, Aviv and Victor Aguirregabiria**, “Recent Developments in Empirical IO: Dynamic Demand and Dynamic Games,” *Recent Developments in Empirical IO: Dynamic Demand and Dynamic Games (December 29, 2010). Northwestern University Center for the Study of Industrial Organization Working Paper*, 2010, (0107).
- Otsu, Taisuke, Martin Pesendorfer, Yuya Takahashi et al.**, “Pooling data across markets in dynamic Markov games,” Technical Report, Suntory and Toyota International Centres for Economics and Related Disciplines, LSE 2015.
- Pakes, Ariel, Michael Ostrovsky, and Steven Berry**, “Simple estimators for the parameters of discrete dynamic games (with entry/exit examples),” *The RAND Journal of Economics*, 2007, *38* (2), 373–399.
- Paula, Aureo De**, “Econometric analysis of games with multiple equilibria,” *Annual Review of Economics*, 2012, (0).
- **and Xun Tang**, “Inference of signs of interaction effects in simultaneous games with incomplete information,” *Econometrica*, 2012, *80* (1), 143–172.

- Pesendorfer, Martin and Philipp Schmidt-Dengler**, “Asymptotic least squares estimators for dynamic games,” *The Review of Economic Studies*, 2008, 75 (3), 901–928.
- **and —** , “Sequential estimation of dynamic discrete games: A comment,” *Econometrica*, 2010, 78 (2), 833–842.
- **and Yuya Takahashi**, “Testing for Equilibrium Multiplicity in Dynamic Markov Games,” 2012.
- Pötscher, Benedikt M**, “Effects of model selection on inference,” *Econometric Theory*, 1991, 7 (02), 163–185.
- Robin, Jean-Marc and Richard J Smith**, “Tests of rank,” *Econometric Theory*, 2000, 16 (02), 151–175.
- Rust, John**, “Optimal replacement of GMC bus engines: An empirical model of Harold Zurcher,” *Econometrica: Journal of the Econometric Society*, 1987, pp. 999–1033.
- Su, Che-Lin and Kenneth L Judd**, “Constrained optimization approaches to estimation of structural models,” *Econometrica*, 2012, 80 (5), 2213–2230.
- Sweeting, A.**, “Dynamic product positioning in differentiated product markets: the effect of fees for musical performance rights on the commercial radio industry,” *Econometrica*, 2013, 81 (5), 1763–1803.
- Xiao, Ruli**, “Identification and estimation of finite action games with multiple equilibria,” 2014.