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On the Characterization of Incentive Compatible Mechanisms in General Quasi-linear Environments*

Yu Chen[†]

Abstract

This paper generalizes the Myerson (1981)'s characterization results of incentive compatible mechanisms and optimal incentive compatible mechanisms in general quasi-linear multi-agency situations with more abstract payoff function forms and constraint sets. Both Bayesian implementation and ex post implementation are addressed. In this paper the characterization results are also applied to comparison and equivalence between EPIC and BIC mechanisms, strategic dominance of centralized mechanisms over decentralized menus, and optimal (full) surplus extraction.

Keywords: Bayesian incentive compatible mechanism, ex post incentive compatible mechanism, quasi-linear environment

JEL Classification: C72 D82 D86

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1 Introduction

Myerson (1981) in his well-known paper on optimal auction design discusses a similar characterization result concerning Bayesian incentive compatible¹ mechanisms in a relatively restricted auction setting, which is a special case of the general quasi-linear multi-agency environment. He also extend the characterization result to optimal Bayesian incentive compatible auction mechanisms.

This paper generalizes the characterization results of IC mechanisms² and optimal IC mechanisms in general quasi-linear multi-agency situations. We try to generalize Myerson's characterization results along two directions. 1) we characterize the incentive compatible mechanisms and optimal incentive compatible mechanisms in a more general model setting, especially in more abstract payoff function forms and constraint sets. 2) our characterization results are within not only the context of Bayesian implementation but also that of Ex Post implementation.

Our analysis allows "full interdependence," that is, every agent's payoff depends on all the agents' assignments and types. Although our model is based on a general interdependence situation, all the results still apply to any of its degenerate version, including the separable assignment rules situation and the private valuation situation, in which ex post implementation will degenerate to dominant strategy implementation.

The characterization results of (optimal) IC Mechanisms help us acquire more concrete knowledge about the (optimal) IC Mechanisms and confine them by monotonicity conditions, integration conditions and IR conditions for each agent with the lowest type. Moreover, those results can transfer the original problems to more straightforward or tractable problems and therefore serve as the tools to analyze certain seemingly complicated issues related to optimal mechanism design. In this paper, we also apply the characterization results to comparison and equivalence between EPIC and BIC mechanisms, strategic dominance of centralized mechanisms over decentralized menus, and optimal (full) surplus extraction.

2 Preliminaries

We consider a multi-agency contracting game in which one principal (PL) moves first, and n agents, indexed by $i \in \mathcal{N} = \{1, \dots, n\}$, follow simultaneously and behave non-cooperatively. Agent i (A_i) has the private type $\theta_i \in \Theta_i = [\underline{\theta}_i, \bar{\theta}_i]$. We write $\theta = (\theta_i)_{i \in \mathcal{N}} \in \Theta = \prod_{i=1}^n \Theta_i$ and $\theta_{-i} =$

$(\theta_j)_{j \in \mathcal{N} \setminus \{i\}} \in \theta_{-i} = \prod_{j \neq i}^n \theta_j$. θ has a joint distribution function $F : \Theta \rightarrow [0, 1]$ and an associated joint density function $f : \Theta \rightarrow \mathbb{R}_+$ with respect to Lebesgue measure. For each $i \in \mathcal{N}$, there are a marginal distribution function $F_i : \Theta_i \rightarrow [0, 1]$ and an associated conditional distribution function of θ_{-i} on θ_i , $F_{-i}(\cdot | \theta_i) : \Theta_{-i} \rightarrow [0, 1]$. F_i has a marginal density function with respect to

¹Actually there is no Bayesian update belief, since Myerson assumes independent types.

²Due to the revelation principle, we can restrict our attention to incentive compatible (direct) mechanisms out of general mechanisms in mechanism design.

Lebesgue measure, $f_i : \Theta_i \rightarrow \mathbb{R}_+$, and $F_{-i}(\cdot|\theta_i)$ has a conditional density function with respect to Lebesgue measure, $f_{-i}(\cdot|\theta_i) : \Theta_{-i} \rightarrow \mathbb{R}_+$. In addition, $f(\theta) = f_{-i}(\theta_{-i}|\theta_i)f_i(\theta_i)$, for all $\theta \in \Theta$.

Contracts consist of two parts: assignments (or say activities, decision rules, alternatives, etc.) and transfers. The end-of-period commonly observable assignments for the agents is $x \in \mathcal{X}$, where $\mathcal{X}$ is a partial-ordered set of a metric space. The end-of-period commonly observable monetary transfer from PL to A_i is $t_i \in \mathbb{R}$. Let Borel measurable function $\mathbf{x} : \Theta \rightarrow \mathcal{X}$ denote assignment rule, which specifies to the agents the assignments for each type report profile of the agents. Let Borel measurable function $\mathbf{t}_i : \Theta \rightarrow \mathbb{R}$ denote transfer rule, which specifies to A_i a transfer for each type report profile of the agents³. We write $\mathbf{t} = \prod_{i=1}^n \mathbf{t}_i$. A **mechanism** a pair of functions $(\mathbf{x}, \mathbf{t})$ satisfying $\mathbf{x}(\theta) \in X \subseteq \mathcal{X}$ and $\mathbf{t}(\theta) \in T \subseteq \mathbb{R}^n$. A certain feasible constraint on contracting is embodied in X and T as the subsets of $\mathcal{X}$ and $\mathbb{R}^n$.

We focus on the classic quasi-linear payoffs. PL's ex post payoff function $U : \mathcal{X} \times \mathbb{R}^n \times \Theta \rightarrow \mathbb{R}$ takes the quasi-linear form:

$$U(x, t, \theta) \equiv u(x, \theta) - \sum_{i=1}^n t_i,$$

where $u : \mathcal{X} \times \Theta \rightarrow \mathbb{R}$ is the payoff function from the agents' assignments. A_i 's ex post payoff function $V_i : \mathcal{X} \times \mathbb{R} \times \Theta \rightarrow \mathbb{R}$ takes the quasi-linear form:

$$V_i(x, t_i, \theta) \equiv v_i(x, \theta) + t_i,$$

where $v_i : \mathcal{X} \times \Theta \rightarrow \mathbb{R}$ is the payoff function from the agents' assignments. v_i is continuous. A_i has the reservation utility $r_i \in \mathbb{R}$.

3 Principal-Agent Problems over Mechanism Design

Under the general quasi-linear environment, there are two major classes of implementation are notable for study, depending on the information structure. The first class is Bayesian implementation in terms of Bayesian incentive compatible mechanism.

A direct mechanism $(\mathbf{x}, \mathbf{t})$ is **Bayesian incentive compatible (BIC)** if for all $i \in \mathcal{N}$ and $\theta_i \in \Theta_i$,

$$\int_{\Theta_{-i}} v_i(\mathbf{x}(\theta), \theta) + \mathbf{t}_i(\theta) dF_{-i}(\theta_{-i}|\theta_i) \geq \int_{\Theta_{-i}} v_i(\mathbf{x}(\theta'_i, \theta_{-i}), \theta) + \mathbf{t}_i(\theta'_i, \theta_{-i}) dF_{-i}(\theta_{-i}|\theta_i),$$

for all $\theta'_i \in \Theta_i$. A BIC mechanism $(\mathbf{x}, \mathbf{t})$ is (interim) **Individual Rational (IR)** if for all $i \in \mathcal{N}$ and $\theta_i \in \Theta_i$,

$$\int_{\Theta_{-i}} v_i(\mathbf{x}(\theta), \theta) + \mathbf{t}_i(\theta) dF_{-i}(\theta_{-i}|\theta_i) \geq r_i.$$

³Note that $\mathbf{t}_i$ must be Lebesgue measurable as well.

We thus present Principal-Agent problem with Bayesian implementation.

$$(\mathbf{P1}) \max_{(\mathbf{x}, \mathbf{t})} \int_{\Theta} u(\mathbf{x}(\theta), \theta) - \sum_{i=1}^n \mathbf{t}_i(\theta) dF(\theta)$$

s.t. $(\mathbf{x}, \mathbf{t})$ is **BIC** and **IR**

For all $\theta \in \Theta, \mathbf{x}(\theta) \in X, \mathbf{t}(\theta) \in T$

The second class is ex post implementation in terms of ex post incentive compatible mechanism. A direct mechanism $(\mathbf{x}, \mathbf{t})$ is **Ex Post incentive compatible (EPIC)** if for all $i \in \mathcal{N}$ and $\theta \in \Theta$,

$$v_i(\mathbf{x}(\theta), \theta) + \mathbf{t}_i(\theta) \geq v_i(\mathbf{x}(\theta'_i, \theta_{-i}), \theta) + \mathbf{t}_i(\theta'_i, \theta_{-i}),$$

for all $\theta'_i \in \Theta_i$. A EPIC mechanism $(\mathbf{x}, \mathbf{t})$ is (ex post) **Individual Rational (IR)** if for all $i \in \mathcal{N}$ and $\theta \in \Theta$,

$$v_i(\mathbf{x}(\theta), \theta) + \mathbf{t}_i(\theta) \geq r_i$$

We thus present Principal-Agent problem with Bayesian implementation.

$$(\mathbf{P2}) \max_{(\mathbf{x}, \mathbf{t})} \int_{\Theta} u(\mathbf{x}(\theta), \theta) - \sum_{i=1}^n \mathbf{t}_i(\theta) dF(\theta)$$

s.t. $(\mathbf{x}, \mathbf{t})$ is **EPIC** and **IR**

For all $\theta \in \Theta, \mathbf{x}(\theta) \in X, \mathbf{t}(\theta) \in T$

We also need some assumptions for our analysis as below.

Assumption 1 $v_i(x, \theta)$ is Borel-measurable in x^4 and differentiable in $\theta_i \in [\underline{\theta}_i, \bar{\theta}_i]$.

Assumption 2 $\frac{\partial v_i(x, \theta)}{\partial \theta_i}$ is continuous in x and θ .

Assumption 3 $u(x, \theta)$ is Borel measurable in x and θ .⁵

Assumption 4 $v_i(x, \theta)$ has the strictly increasing difference (stronger than single crossing property) in x and θ_i , for all $\theta_{-i} \in \Theta_{-i}$, that is,

$$\forall x'' > x' \text{ in } \mathcal{X}, b_i > a_i \text{ in } \Theta_i,$$

$$v_i(x'', b_i, \theta_{-i}) - v_i(x', b_i, \theta_{-i}) > v_i(x'', a_i, \theta_{-i}) - v_i(x', a_i, \theta_{-i})$$

Or equivalently, $\frac{\partial v_i(x, \theta)}{\partial \theta_i}$ is strictly increasing in x .

Moreover, we define

$$\bar{V}_i(\theta; \mathbf{x}, \mathbf{t}_i) = v_i(\mathbf{x}(\theta), \theta) + \mathbf{t}_i(\theta)$$

as A_i 's ex post equilibrium payoff with respect to direct mechanism $(\mathbf{x}, \mathbf{t})$ when he truthfully reports his type θ_i given other agents' truthful type reports θ_{-i} . We also define

$$\hat{V}_i(\theta_i; \mathbf{x}, \mathbf{t}_i) = \int_{\Theta_{-i}} \bar{V}_i(\theta; \mathbf{x}, \mathbf{t}_i) dF_{-i}(\theta_{-i} | \theta_i) = \int_{\Theta_{-i}} v_i(\mathbf{x}(\theta), \theta) + \mathbf{t}_i(\theta) dF_{-i}(\theta_{-i} | \theta_i)$$

as A_i 's interim (Bayesian) equilibrium payoff with respect to direct mechanism $(\mathbf{x}, \mathbf{t})$ when he truthfully reports his type θ_i .

⁴To be more specific, we can assume $v_i(x, \theta)$ is continuous in x .

⁵To be more specific, we can assume $u(x, \theta)$ is continuous in x and θ .

4 Characterization Results of IC Mechanisms

4.1 Characterization of BIC Mechanisms

It is difficult to generalize Myerson's Characterization result in a general Bayesian information structure. We instead address a particular but still important case of information structure.

Definition 1 *There are no substantive Bayesian update beliefs in the information structure of mechanism design if in the information structure on which PL and the agents rely, $F_{-i}(\theta_{-i}|\theta_i) \equiv F_{-i}(\theta_{-i}|\theta'_i) \equiv F_{-i}(\theta_{-i}), \forall \theta_{-i}, \theta_i, \theta'_i$. Moreover, $f_{-i}(\theta_{-i}|\theta_i) \equiv f_{-i}(\theta_{-i}|\theta'_i) \equiv f_{-i}(\theta_{-i}), \forall \theta_{-i}, \theta_i, \theta'_i$.*

If θ is independently distributed, then θ_i has a distribution function $F_i : \Theta_i \rightarrow [0, 1]$ and an associated density function with respect to Lebesgue measure $f_i : \Theta_i \rightarrow \mathbb{R}_+$. There must not be substantive Bayesian update beliefs in the information structure. In this case, we can provide a generalized result of characterization of BIC mechanisms with more straightforward and economically intuitive conditions.

Proposition 1 *Suppose that there are no substantive Bayesian update beliefs. Under assumptions 1 and 2, a direct mechanism $(\mathbf{x}, \mathbf{t})$ is **BIC and IR** iff for each $i \in \mathcal{N}$,*

$$\int_{\Theta_{-i}} \frac{\partial v_i(\mathbf{x}(h, \theta_{-i}), \theta)}{\partial \theta_i} dF_{-i}(\theta_{-i}) \text{ is increasing in } h \text{ on } \Theta_i, \quad (1)$$

$$\widehat{V}_i(b_i; \mathbf{x}, \mathbf{t}_i) - \widehat{V}_i(a_i; \mathbf{x}, \mathbf{t}_i) = \int_{a_i}^{b_i} \int_{\Theta_{-i}} \left[\frac{\partial v_i(\mathbf{x}(\theta), \theta)}{\partial \theta_i} \right] dF_{-i}(\theta_{-i}) d\theta_i, \forall a_i, b_i \in \Theta_i,^6 \quad (2)$$

$$\widehat{V}_i(\underline{\theta}_i; \mathbf{x}, \mathbf{t}_i) \geq r_i. \quad (3)$$

Proof. " $\implies$ "BIC condition implies that $\forall a_i < b_i \in \Theta_i$,

$$\begin{aligned} \widehat{V}_i(b_i; \mathbf{x}, \mathbf{t}_i) &\geq \int_{\Theta_{-i}} v_i(\mathbf{x}(a_i, \theta_{-i}), b_i, \theta_{-i}) + \mathbf{t}_i(a_i, \theta_{-i}) dF_{-i}(\theta_{-i}) \\ &= \int_{\Theta_{-i}} v_i(\mathbf{x}(a_i, \theta_{-i}), a_i, \theta_{-i}) + \mathbf{t}_i(a_i, \theta_{-i}) dF_{-i}(\theta_{-i}) \\ &\quad + \int_{\Theta_{-i}} v_i(\mathbf{x}(a_i, \theta_{-i}), b_i, \theta_{-i}) dF_{-i}(\theta_{-i}) - \int_{\Theta_{-i}} v_i(\mathbf{x}(a_i, \theta_{-i}), a_i, \theta_{-i}) dF_{-i}(\theta_{-i}) \\ &= \widehat{V}_i(a_i; \mathbf{x}, \mathbf{t}_i) \\ &\quad + \int_{\Theta_{-i}} v_i(\mathbf{x}(a_i, \theta_{-i}), b_i, \theta_{-i}) dF_{-i}(\theta_{-i}) - \int_{\Theta_{-i}} v_i(\mathbf{x}(a_i, \theta_{-i}), a_i, \theta_{-i}) dF_{-i}(\theta_{-i}) \end{aligned}$$

Thus,

$$\begin{aligned} &\widehat{V}_i(b_i; \mathbf{x}, \mathbf{t}_i) - \widehat{V}_i(a_i; \mathbf{x}, \mathbf{t}_i) \\ &\geq \int_{\Theta_{-i}} v_i(\mathbf{x}(a_i, \theta_{-i}), b_i, \theta_{-i}) dF_{-i}(\theta_{-i}) - \int_{\Theta_{-i}} v_i(\mathbf{x}(a_i, \theta_{-i}), a_i, \theta_{-i}) dF_{-i}(\theta_{-i}) \end{aligned} \quad (4)$$

Reversing b_i and a_i in (4) and combining with (5), we have

$$\begin{aligned}
& \int_{\Theta_{-i}} v_i(\mathbf{x}(a_i, \theta_{-i}), b_i, \theta_{-i}) dF_{-i}(\theta_{-i}) - \int_{\Theta_{-i}} v_i(\mathbf{x}(a_i, \theta_{-i}), a_i, \theta_{-i}) dF_{-i}(\theta_{-i}) \\
& \leq \widehat{V}_i(b_i; \mathbf{x}, \mathbf{t}_i) - \widehat{V}_i(a_i; \mathbf{x}, \mathbf{t}_i) \\
& \leq \int_{\Theta_{-i}} v_i(\mathbf{x}(b_i, \theta_{-i}), b_i, \theta_{-i}) dF_{-i}(\theta_{-i}) - \int_{\Theta_{-i}} v_i(\mathbf{x}(b_i, \theta_{-i}), a_i, \theta_{-i}) dF_{-i}(\theta_{-i}) \quad (5)
\end{aligned}$$

Hence,

$$\begin{aligned}
& \int_{\Theta_{-i}} v_i(\mathbf{x}(a_i, \theta_{-i}), b_i, \theta_{-i}) dF_{-i}(\theta_{-i}) - \int_{\Theta_{-i}} v_i(\mathbf{x}(a_i, \theta_{-i}), a_i, \theta_{-i}) dF_{-i}(\theta_{-i}) \\
& \leq \int_{\Theta_{-i}} v_i(\mathbf{x}(b_i, \theta_{-i}), b_i, \theta_{-i}) dF_{-i}(\theta_{-i}) - \int_{\Theta_{-i}} v_i(\mathbf{x}(b_i, \theta_{-i}), a_i, \theta_{-i}) dF_{-i}(\theta_{-i}) \quad (6)
\end{aligned}$$

Now suppose (1) does not hold, then for some $b'_i > a'_i$,

$$\int_{\Theta_{-i}} \frac{\partial v_i(\mathbf{x}(b'_i, \theta), \theta_i, \theta_{-i})}{\partial \theta_i} dF_{-i}(\theta_{-i}) < \int_{\Theta_{-i}} \frac{\partial v_i(\mathbf{x}(a'_i, \theta), \theta_i, \theta_{-i})}{\partial \theta_i} dF_{-i}(\theta_{-i})$$

Thus,

$$\begin{aligned}
& \int_{a'_i}^{b'_i} \int_{\Theta_{-i}} \frac{\partial v_i(\mathbf{x}(b'_i, \theta), \theta_i, \theta_{-i})}{\partial \theta_i} dF_{-i}(\theta_{-i}) d\theta_i < \int_{a'_i}^{b'_i} \int_{\Theta_{-i}} \frac{\partial v_i(\mathbf{x}(a'_i, \theta), \theta_i, \theta_{-i})}{\partial \theta_i} dF_{-i}(\theta_{-i}) d\theta_i \\
& \int_{\Theta_{-i}} \int_{a_i}^{b_i} \frac{\partial v_i(\mathbf{x}(b'_i, \theta), \theta_i, \theta_{-i})}{\partial \theta_i} d\theta_i dF_{-i}(\theta_{-i}) < \int_{\Theta_{-i}} \int_{a'_i}^{b'_i} \frac{\partial v_i(\mathbf{x}(a'_i, \theta), \theta_i, \theta_{-i})}{\partial \theta_i} d\theta_i dF_{-i}(\theta_{-i}) \\
& \int_{\Theta_{-i}} v_i(\mathbf{x}(b'_i, \theta_{-i}), b'_i, \theta_{-i}) - v_i(\mathbf{x}(b'_i, \theta_{-i}), a'_i, \theta_{-i}) dF_{-i}(\theta_{-i}) < \int_{\Theta_{-i}} v_i(\mathbf{x}(a'_i, \theta_{-i}), b'_i, \theta_{-i}) - v_i(\mathbf{x}(a'_i, \theta_{-i}), a'_i, \theta_{-i}) dF_{-i}(\theta_{-i})
\end{aligned}$$

This contradicts (7). Therefore (1) follows.

Next take the limit in (6) as $b_i \rightarrow a_i$, (6) implies that

$$\frac{\partial \widehat{V}_i(\theta_i; \mathbf{x}, \mathbf{t}_i)}{\partial \theta_i} = \int_{\Theta_{-i}} \frac{\partial v_i(\mathbf{x}(\theta), \theta_i, \theta_{-i})}{\partial \theta_i} dF_{-i}(\theta_{-i}) \quad (7)$$

This implies that $\widehat{V}_i(\theta_i; \mathbf{x}, \mathbf{t}_i)$ is differentiable in $[\underline{\theta}_i, \bar{\theta}_i]$. By assumption 2 and hypothesis (1), $\int_{\Theta_{-i}} \frac{\partial v_i(\mathbf{x}(\theta), \theta_i, \theta_{-i})}{\partial \theta_i} dF_{-i}(\theta_{-i})$ can only have a countable number of discontinuities, in other words, it is continuous in θ_i almost everywhere with respect to Lebesgue measure. Moreover, $\int_{\Theta_{-i}} \frac{\partial v_i(\mathbf{x}(\theta), \theta_i, \theta_{-i})}{\partial \theta_i} dF_{-i}(\theta_{-i})$ is bounded on $X \times \Theta$. Hence, by Rudin theorem 6.10. $\frac{\partial \widehat{V}_i(\theta_i; \mathbf{x}, \mathbf{t}_i)}{\partial \theta_i}$ is Riemann integrable with respect to θ_i .⁷ Then the fundamental theorem of calculus (Rudin 6.21

⁷In this case, by Aliprantis and Border Theorem 11.32, Riemann integrability implies Lebesgue integrability. Riemann integral will coincide with Lebesgue integral.

P134) applies. We have

$$\widehat{V}_i(b_i; \mathbf{x}, \mathbf{t}_i) - \widehat{V}_i(a_i; \mathbf{x}, \mathbf{t}_i) = \int_{a_i}^{b_i} \int_{\Theta_{-i}} \frac{\partial v_i(\mathbf{x}(\theta), \theta_i, \theta_{-i})}{\partial \theta_i} dF_{-i}(\theta_{-i}) d\theta_i, \forall a_i < b_i \in \Theta_i,$$

Thus, (2) follows.

Lastly, (3) directly follows from **IR** condition.

" $\implies$ "First since $\frac{\partial v_i(x, \theta)}{\partial \theta_i} \geq 0$, (2) and (3) imply $\widehat{V}_i(\theta_i; \mathbf{x}, \mathbf{t}_i) \geq r_i, \forall \theta_i \in \Theta_i$. So IR condition holds. Note that **BIC** condition is equivalent to (4) indeed. Now we need to show that (4) holds.

Suppose (4) does not hold. Then, for some j, b_j, a_j ,

$$\begin{aligned} & \int_{\Theta_{-j}} v_j(\mathbf{x}(a_j, \theta_{-j}), b_j, \theta_{-j}) dF_{-j}(\theta_{-j}) - \int_{\Theta_{-j}} v_j(\mathbf{x}(a_j, \theta_{-j}), a_j, \theta_{-j}) dF_{-j}(\theta_{-j}) \\ & < \widehat{V}_j(b_j; \mathbf{x}, \mathbf{t}_j) - \widehat{V}_j(a_j; \mathbf{x}, \mathbf{t}_j) = \int_{a_j}^{b_j} \int_{\Theta_{-j}} \frac{\partial v_j(\mathbf{x}(\theta), \theta_j, \theta_{-j})}{\partial \theta_j} dF_{-j}(\theta_{-j}) d\theta_j \end{aligned} \quad (8)$$

Then

$$\begin{aligned} & \int_{a_j}^{b_j} \int_{\Theta_{-j}} \frac{\partial v_j(\mathbf{x}(\theta), \theta_j, \theta_{-j})}{\partial \theta_j} dF_{-j}(\theta_{-j}) d\theta_j \\ & < \int_{\Theta_{-j}} v_j(\mathbf{x}(a_j, \theta_{-j}), b_j, \theta_{-j}) dF_{-j}(\theta_{-j}) - \int_{\Theta_{-j}} v_j(\mathbf{x}(a_j, \theta_{-j}), a_j, \theta_{-j}) dF_{-j}(\theta_{-j}) \\ & = \int_{\Theta_{-j}} [v_j(\mathbf{x}(a_j, \theta_{-j}), b_j, \theta_{-j}) - v_j(\mathbf{x}(a_j, \theta_{-j}), a_j, \theta_{-j})] dF_{-j}(\theta_{-j}) \\ & = \int_{\Theta_{-j}} \int_{a_j}^{b_j} \frac{\partial v_j(\mathbf{x}(a_j, \theta_{-j}), \theta_j, \theta_{-j})}{\partial \theta_j} d\theta_j dF_{-j}(\theta_{-j}) \\ & = \int_{a_j}^{b_j} \int_{\Theta_{-j}} \frac{\partial v_j(\mathbf{x}(a_j, \theta_{-j}), \theta_j, \theta_{-j})}{\partial \theta_j} dF_{-j}(\theta_{-j}) d\theta_j \end{aligned}$$

But (1) implies this is not possible. So this contradiction completes the proof. ■

4.2 Characterization of EPIC Mechanisms

Since EPIC mechanisms do not require Bayesian update beliefs in play, it does not matter whether types are independent or not in ex post implementation.

Proposition 2 Under assumptions 1-3, $(\mathbf{x}, \mathbf{t})$ is **EPIC and IR** iff for each $i \in \mathcal{N}$, $\forall \theta_{-i} \in \Theta_{-i}$,

$$\frac{\partial v_i(\mathbf{x}(h, \theta_{-i}), \theta)}{\partial \theta_i} \text{ is increasing in } h \text{ on } \Theta_i, \quad (9)$$

$$\bar{V}_i(b_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) - \bar{V}_i(a_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) = \int_{a_i}^{b_i} \frac{\partial v_i(\mathbf{x}(\theta), \theta)}{\partial \theta_i} d\theta_i, \forall a_i, b_i \in \Theta_i,^8 \quad (10)$$

$$\bar{V}_i(\theta_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) \geq r_i. \quad (11)$$

Proof. " $\implies$ "EPIC condition implies that $\forall a_i < b_i \in \Theta_i, \forall \theta_{-i} \in \Theta_{-i}$,

$$\begin{aligned} \bar{V}_i(b_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) &\geq v_i(\mathbf{x}(a_i, \theta_{-i}), b_i, \theta_{-i}) + \mathbf{t}_i(a_i, \theta_{-i}) \\ &= v_i(\mathbf{x}(a_i, \theta_{-i}), a_i, \theta_{-i}) + \mathbf{t}_i(a_i, \theta_{-i}) \\ &\quad + v_i(\mathbf{x}(a_i, \theta_{-i}), b_i, \theta_{-i}) - v_i(\mathbf{x}(a_i, \theta_{-i}), a_i, \theta_{-i}) \\ &= \bar{V}_i(a_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) \\ &\quad + v_i(\mathbf{x}(a_i, \theta_{-i}), b_i, \theta_{-i}) - v_i(\mathbf{x}(a_i, \theta_{-i}), a_i, \theta_{-i}) \end{aligned}$$

Thus,

$$\begin{aligned} &\bar{V}_i(b_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) - \bar{V}_i(a_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) \\ &\geq v_i(\mathbf{x}(a_i, \theta_{-i}), b_i, \theta_{-i}) - v_i(\mathbf{x}(a_i, \theta_{-i}), a_i, \theta_{-i}) \end{aligned} \quad (12)$$

Reversing b_i and a_i and combining (), we have

$$\begin{aligned} &v_i(\mathbf{x}(a_i, \theta_{-i}), b_i, \theta_{-i}) - v_i(\mathbf{x}(a_i, \theta_{-i}), a_i, \theta_{-i}) \\ &\leq \bar{V}_i(b_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) - \bar{V}_i(a_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) \\ &\leq v_i(\mathbf{x}(b_i, \theta_{-i}), b_i, \theta_{-i}) - v_i(\mathbf{x}(b_i, \theta_{-i}), a_i, \theta_{-i}) \end{aligned} \quad (13)$$

Thus,

$$\begin{aligned} &v_i(\mathbf{x}(a_i, \theta_{-i}), b_i, \theta_{-i}) - v_i(\mathbf{x}(a_i, \theta_{-i}), a_i, \theta_{-i}) \\ &\leq v_i(\mathbf{x}(b_i, \theta_{-i}), b_i, \theta_{-i}) - v_i(\mathbf{x}(b_i, \theta_{-i}), a_i, \theta_{-i}) \end{aligned} \quad (14)$$

Now suppose (9) does not hold, then $\forall \theta_{-i} \in \Theta_{-i}$, for some $b'_i > a'_i$

$$\frac{\partial v_i(\mathbf{x}(b'_i, \theta), \theta_i, \theta_{-i})}{\partial \theta_i} < \frac{\partial v_i(\mathbf{x}(a'_i, \theta), \theta_i, \theta_{-i})}{\partial \theta_i}$$

Thus,

$$\begin{aligned} \int_{a'_i}^{b'_i} \frac{\partial v_i(\mathbf{x}(b'_i, \theta), \theta_i, \theta_{-i})}{\partial \theta_i} d\theta_i &< \int_{a'_i}^{b'_i} \frac{\partial v_i(\mathbf{x}(a'_i, \theta), \theta_i, \theta_{-i})}{\partial \theta_i} d\theta_i \\ v_i(\mathbf{x}(b'_i, \theta_{-i}), b'_i, \theta_{-i}) - v_i(\mathbf{x}(b'_i, \theta_{-i}), a'_i, \theta_{-i}) &< v_i(\mathbf{x}(a'_i, \theta_{-i}), b'_i, \theta_{-i}) - v_i(\mathbf{x}(a'_i, \theta_{-i}), a'_i, \theta_{-i}) \end{aligned}$$

This contradicts (14). Therefore (9) follows.

Next take the limit in (13) as $b_i \rightarrow a_i$, it implies that

$$\frac{\partial \bar{V}_i(\theta_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i)}{\partial \theta_i} = \frac{\partial v_i(\mathbf{x}(\theta), \theta_i, \theta_{-i})}{\partial \theta_i} \quad (15)$$

This implies that $\bar{V}_i(\theta_i; \mathbf{x}, \mathbf{t}_i)$ is differentiable in $[\underline{\theta}_i, \bar{\theta}_i]$. Then by similar argument in the proof of Proposition 1,

$$\bar{V}_i(b_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) - \bar{V}_i(a_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) = \int_{a_i}^{b_i} \frac{\partial v_i(\mathbf{x}(\theta), \theta_i, \theta_{-i})}{\partial \theta_i} d\theta_i, \forall a_i, b_i \in \Theta_i,$$

Therefore, (10) is satisfied.

Lastly, (11) directly follows from **IR** condition.

" $\Leftarrow$ " First since $\frac{\partial v_i(x, \theta)}{\partial \theta_i} \geq 0$, (10) and (11) imply $\bar{V}_i(\theta_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) \geq r_i, \forall \theta \in \Theta$. So IR condition holds.

Note that **EPIC** condition is equivalent to (14) indeed. Now we need to show that (14) holds.

Suppose (14) does not hold. Then, for some $j, b_j, a_j, \hat{\theta}_{-j}$,

$$\begin{aligned} &v_j(\mathbf{x}(a_j, \hat{\theta}_{-j}), b_j, \hat{\theta}_{-j}) - v_j(\mathbf{x}(a_j, \hat{\theta}_{-j}), a_j, \hat{\theta}_{-j}) \\ &< \bar{V}_j(b_j, \hat{\theta}_{-j}; \mathbf{x}, \mathbf{t}_j) - \bar{V}_j(a_j, \hat{\theta}_{-j}; \mathbf{x}, \mathbf{t}_j) \\ &= \int_{a_j}^{b_j} \frac{\partial v_j(\mathbf{x}(\theta_j, \hat{\theta}_{-j}), \theta_j, \hat{\theta}_{-j})}{\partial \theta_j} d\theta_j \end{aligned}$$

Thus,

$$\begin{aligned} &\int_{a_j}^{b_j} \frac{\partial v_j(\mathbf{x}(\theta), \theta_j, \hat{\theta}_{-j})}{\partial \theta_j} d\theta_j \\ &< v_j(\mathbf{x}(a_j, \hat{\theta}_{-j}), b_j, \hat{\theta}_{-j}) - v_j(\mathbf{x}(a_j, \hat{\theta}_{-j}), a_j, \hat{\theta}_{-j}) \\ &= \int_{a_j}^{b_j} \frac{\partial v_j(\mathbf{x}(a_j, \hat{\theta}_{-j}), \theta_j, \hat{\theta}_{-j})}{\partial \theta_j} d\theta_j \end{aligned}$$

But $\frac{\partial v_j(x, \theta)}{\partial \theta_j} \geq 0$, and (9) implies this is not possible. So this contradiction completes the proof.

■

4.3 Interpretation of Characterization Conditions

Condition 1 (respectively, 9) is a monotonicity condition. It shows that in the BIC (respectively, EPIC) mechanism, the assignment rule must be such that each A_i 's interim expected (respectively, ex post) marginal payoff from assignments with respect to A_i 's type will increase as A_i 's type report increases for the assignment rule. This is a constraint solely on the assignment rules.

Condition 2 (respectively, 10) is an integration condition. It shows that any difference between A_i 's interim (respectively, ex post) equilibrium payoffs causing by different A_i 's types will be equal to the integral of interim expected marginal payoff (respectively, marginal payoff) from assignments with respect to A_i 's type within the two types.

Condition 2 (respectively, 10) actually corresponds to the envelope theorem as mentioned in Milgrom and Segal (2002). Note that Lebesgue measure is invariant in translation on a closed interval in a Euclidean space. Clearly, condition 2 is equivalent to

$$\widehat{V}_i(\theta_i; \mathbf{x}, \mathbf{t}_i) = \widehat{V}_i(\underline{\theta}_i; \mathbf{x}, \mathbf{t}_i) + \int_{\underline{\theta}_i}^{\theta_i} \int_{\Theta_{-i}} \frac{\partial v_i(\mathbf{x}(\theta), \theta)}{\partial \theta_i} dF_{-i}(\theta_{-i}) d\theta_i,$$

for each $\theta_i \in \Theta_i$. Condition 10 is equivalent to

$$\overline{V}_i(\theta_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) = \overline{V}_i(\underline{\theta}_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) + \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}(\theta), \theta)}{\partial \theta_i} d\theta_i,$$

for each $\theta \in \Theta$. Hence, under any BIC (respectively, EPIC) mechanism, we can know A_i 's Bayesian (respectively ex post) equilibrium payoff mapping with respect to his own type serving as the parameter.

Moreover, equation (7) (respectively (15)), as the corresponding differentiation condition, uncovers how A_i 's Bayesian (respectively ex post) equilibrium payoff will change with respect to his own type.

It is worth noting that condition 3 (respectively 11) is only for IR condition. If we only characterize BIC mechanisms, we can omit condition 3 (respectively 11). Condition 3 (respectively 11), joint with the other two conditions, simplifies the original IR condition. It is only a IR condition for each agent with the lowest type. As long as each agent with the lowest type must be willing to participate, each agent with any other types will always participate.

4.4 Recoverability of Transfer Rules

Proposition 3 *Under assumptions 1 and 2, condition (2) is equivalent to*

$$\begin{aligned} \int_{\Theta_{-i}} \mathbf{t}_i(\theta_i, \theta_{-i}) dF_{-i}(\theta_{-i}) &= \int_{\Theta_{-i}} \left[\int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}(h, \theta_{-i}), h, \theta_{-i})}{\partial \theta_i} dh - v_i(\mathbf{x}(\theta_i, \theta_{-i}), \theta) \right. \\ &\quad \left. + v_i(\mathbf{x}(\underline{\theta}_i, \theta_{-i}), \underline{\theta}_i, \theta_{-i}) + \mathbf{t}_i(\underline{\theta}_i, \theta_{-i}) \right] dF_{-i}(\theta_{-i}). \end{aligned} \quad (16)$$

Proof. Given BIC $(\mathbf{x}, \mathbf{t})$,

$$\begin{aligned}
\widehat{V}_i(\theta_i; \mathbf{x}, \mathbf{t}_i) - \widehat{V}_i(\underline{\theta}_i; \mathbf{x}, \mathbf{t}_i) &= \int_{\underline{\theta}_i}^{\theta_i} \int_{\Theta_{-i}} \frac{\partial v_i(\mathbf{x}(h, \theta_{-i}), h, \theta_{-i})}{\partial \theta_i} dF_{-i}(\theta_{-i}) dh \\
&\implies \int_{\Theta_{-i}} [v_i(\mathbf{x}(\theta_i, \theta_{-i}), \theta) + \mathbf{t}_i(\theta_i, \theta_{-i})] dF_{-i}(\theta_{-i}) \\
&= \int_{\Theta_{-i}} [v_i(\mathbf{x}(\underline{\theta}_i, \theta_{-i}), \underline{\theta}_i, \theta_{-i}) + \mathbf{t}_i(\underline{\theta}_i, \theta_{-i})] dF_{-i}(\theta_{-i}) \\
&\quad + \int_{\underline{\theta}_i}^{\theta_i} \int_{\Theta_{-i}} \frac{\partial v_i(\mathbf{x}(h, \theta_{-i}), h, \theta_{-i})}{\partial \theta_i} dF_{-i}(\theta_{-i}) dh \\
&\implies \int_{\Theta_{-i}} \mathbf{t}_i(\theta_i, \theta_{-i}) dF_{-i}(\theta_{-i}) \\
&= \int_{\Theta_{-i}} \left[\int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}(h, \theta_{-i}), h, \theta_{-i})}{\partial \theta_i} dh - v_i(\mathbf{x}(\theta_i, \theta_{-i}), \theta) \right. \\
&\quad \left. + v_i(\mathbf{x}(\underline{\theta}_i, \theta_{-i}), \underline{\theta}_i, \theta_{-i}) + \mathbf{t}_i(\underline{\theta}_i, \theta_{-i}) \right] dF_{-i}(\theta_{-i}).
\end{aligned}$$

Also note that Lebesgue measure is invariant in translation on a closed interval. So condition (2) is equivalent to (16). ■

This proposition suggests that in BIC implementation transfer rules can be recovered from assignment rules. If mechanism $(\mathbf{x}, \mathbf{t})$ is BIC, let $\varepsilon(\theta_{-i}) = v_i(\mathbf{x}(\underline{\theta}_i, \theta_{-i}), \underline{\theta}_i, \theta_{-i}) + \mathbf{t}_i(\underline{\theta}_i, \theta_{-i})$. Then,

$$\int_{\Theta_{-i}} \mathbf{t}_i(\theta) dF_{-i}(\theta_{-i}) = \int_{\Theta_{-i}} \left[\int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}(h, \theta_{-i}), h, \theta_{-i})}{\partial \theta_i} dh - v_i(\mathbf{x}(\theta_i, \theta_{-i}), \theta) + \varepsilon(\theta_{-i}) \right] dF_{-i}(\theta_{-i})$$

for some real-valued function $\varepsilon(\theta_{-i})$.

Proposition 4 *Under assumptions 1 and 2, condition (10) is equivalent to*

$$\mathbf{t}_i(\theta_i, \theta_{-i}) = \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}(h, \theta_{-i}), h, \theta_{-i})}{\partial \theta_i} dh - v_i(\mathbf{x}(\theta_i, \theta_{-i}), \theta) + v_i(\mathbf{x}(\underline{\theta}_i, \theta_{-i}), \underline{\theta}_i, \theta_{-i}) + \mathbf{t}_i(\underline{\theta}_i, \theta_{-i}). \tag{17}$$

Proof. By condition (10)

$$\begin{aligned}
\overline{V}_i(\theta_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) - \overline{V}_i(\underline{\theta}_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) &= \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}(h, \theta_{-i}), h, \theta_{-i})}{\partial \theta_i} dh \\
v_i(\mathbf{x}(\theta_i, \theta_{-i}), \theta) + \mathbf{t}_i(\theta_i, \theta_{-i}) &= v_i(\mathbf{x}(\underline{\theta}_i, \theta_{-i}), \underline{\theta}_i, \theta_{-i}) + \mathbf{t}_i(\underline{\theta}_i, \theta_{-i}) + \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}(h, \theta_{-i}), h, \theta_{-i})}{\partial \theta_i} dh \\
\mathbf{t}_i(\theta_i, \theta_{-i}) &= \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}(h, \theta_{-i}), h, \theta_{-i})}{\partial \theta_i} dh - v_i(\mathbf{x}(\theta_i, \theta_{-i}), \theta) \\
&\quad + v_i(\mathbf{x}(\underline{\theta}_i, \theta_{-i}), \underline{\theta}_i, \theta_{-i}) + \mathbf{t}_i(\underline{\theta}_i, \theta_{-i}).
\end{aligned}$$

Also note that Lebesgue measure is invariant in translation on a closed interval. So condition (10) is equivalent to (17). ■

This proposition suggests that in BIC implementation transfer rules can also be recovered from assignment rules. Let $\eta(\theta_{-i}) = v_i(\mathbf{x}(\underline{\theta}_i, \theta_{-i}), \underline{\theta}_i, \theta_{-i}) + \mathbf{t}_i(\underline{\theta}_i, \theta_{-i})$. If a mechanism $(\mathbf{x}, \mathbf{t})$ is EPIC, then

$$\mathbf{t}_i(\theta_i, \theta_{-i}) = \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}(h, \theta_{-i}), h, \theta_{-i})}{\partial \theta_i} dh - v_i(\mathbf{x}(\theta_i, \theta_{-i}), \theta) + \eta(\theta_{-i})$$

for some real-valued function $\eta(\theta_{-i})$.

4.5 Characterization of Optimal Mechanisms

Based on characterization results of incentive compatible mechanisms, we can further characterize the optimal incentive compatible mechanisms.

4.5.1 Characterization of Optimal BIC Mechanisms

Proposition 5 *Suppose that there are no substantive Bayesian update beliefs. Under assumptions 1-3, $(\mathbf{x}^*, \mathbf{t}^*)$ is the **optimal mechanism solving P1** if*

$$\begin{aligned} (\mathbf{x}^*, \mathbf{t}^*) &\in \arg \max_{\mathbf{x}, \mathbf{t}} \int_{\Theta} \{u(\mathbf{x}(\theta), \theta) + \sum_{i=1}^n (v_i(\mathbf{x}(\theta), \theta) \\ &\quad - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial v_i(\mathbf{x}(\theta), \theta_i, \theta_{-i})}{\partial \theta_i}) - \sum_{i=1}^n \widehat{V}_i(\underline{\theta}_i; \mathbf{x}, \mathbf{t}_i)\} dF(\theta) \\ &\quad s.t. \int_{\Theta_{-i}} \frac{\partial v_i(\mathbf{x}(h, \theta_{-i}), \theta_i, \theta_{-i})}{\partial \theta_i} dF_{-i}(\theta_{-i}) \text{ is increasing in } h \text{ on } \Theta_i, \\ \widehat{V}_i(b_i; \mathbf{x}, \mathbf{t}_i) - \widehat{V}_i(a_i; \mathbf{x}, \mathbf{t}_i) &= \int_{a_i}^{b_i} \int_{\Theta_{-i}} [\frac{\partial v_i(\mathbf{x}(\theta), \theta)}{\partial \theta_i}] dF_{-i}(\theta_{-i}) d\theta_i, \forall a_i, b_i \in \Theta_i, \\ \widehat{V}_i(\underline{\theta}_i; \mathbf{x}, \mathbf{t}_i) &\geq r_i, \\ \text{and } \forall \theta &\in \Theta, \mathbf{x}(\theta) \in X, \mathbf{t}(\theta) \in T \end{aligned} \tag{18}$$

Proof. We can rewrite PL's objective function as

$$\begin{aligned} \widehat{U}(\mathbf{x}, \mathbf{t}) &= \int_{\Theta} u(\mathbf{x}(\theta), \theta) - \sum_{i=1}^n \mathbf{t}_i(\theta) dF(\theta) \\ &= \int_{\Theta} u(\mathbf{x}(\theta), \theta) dF(\theta) + \int_{\Theta} \sum_{i=1}^n v_i(\mathbf{x}(\theta), \theta) dF(\theta) - \int_{\Theta} \sum_{i=1}^n v_i(\mathbf{x}(\theta), \theta) + \mathbf{t}_i(\theta) dF(\theta) \end{aligned}$$

By proposition 1,

$$\begin{aligned}
& \int_{\Theta} v_i(\mathbf{x}(\theta), \theta) + \mathbf{t}_i(\theta) dF(\theta) \\
&= \int_{\Theta_i} \widehat{V}_i(\theta_i; \mathbf{x}, \mathbf{t}_i) dF_i(\theta_i) \\
&= \int_{\Theta_i} [\widehat{V}_i(\underline{\theta}_i; \mathbf{x}, \mathbf{t}_i) + \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial \widehat{V}_i(s_i; \mathbf{x}, \mathbf{t}_i)}{\partial \theta_i} ds_i] dF_i(\theta_i) \\
&= \widehat{V}_i(\underline{\theta}_i; \mathbf{x}, \mathbf{t}_i) + \int_{\Theta_i} \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial \widehat{V}_i(s_i; \mathbf{x}, \mathbf{t}_i)}{\partial \theta_i} ds_i f_i(\theta_i) d\theta_i \\
&= \widehat{V}_i(\underline{\theta}_i; \mathbf{x}, \mathbf{t}_i) + \int_{\Theta_i} \int_{s_i}^{\bar{\theta}_i} \frac{\partial \widehat{V}_i(s_i; \mathbf{x}, \mathbf{t}_i)}{\partial \theta_i} f_i(\theta_i) d\theta_i ds_i \quad (\text{interchange the order of integral.}) \\
&= \widehat{V}_i(\underline{\theta}_i; \mathbf{x}, \mathbf{t}_i) + \int_{\Theta_i} (1 - F_i(s_i)) \frac{\partial \widehat{V}_i(s_i; \mathbf{x}, \mathbf{t}_i)}{\partial \theta_i} ds_i \\
&= \widehat{V}_i(\underline{\theta}_i; \mathbf{x}, \mathbf{t}_i) + \int_{\Theta_i} (1 - F_i(s_i)) \int_{\Theta_{-i}} \frac{\partial v_i(\mathbf{x}(s_i, \theta_{-i}), s_i, \theta_{-i})}{\partial \theta_i} dF_{-i}(\theta_{-i}) ds_i \\
&= \widehat{V}_i(\underline{\theta}_i; \mathbf{x}, \mathbf{t}_i) + \int_{\Theta} (1 - F_i(\theta_i)) \frac{\partial v_i(\mathbf{x}(\theta), \theta_i, \theta_{-i})}{\partial \theta_i} f_{-i}(\theta_{-i}) d\theta \\
&= \widehat{V}_i(\underline{\theta}_i; \mathbf{x}, \mathbf{t}_i) + \int_{\Theta} \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial v_i(\mathbf{x}(\theta), \theta_i, \theta_{-i})}{\partial \theta_i} f(\theta) d\theta
\end{aligned}$$

So,

$$\widehat{U}(\mathbf{x}, \mathbf{t}) = \int_{\Theta} \left\{ u(\mathbf{x}(\theta), \theta) + \sum_{i=1}^n \left(v_i(\mathbf{x}(\theta), \theta) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial v_i(\mathbf{x}(\theta), \theta_i, \theta_{-i})}{\partial \theta_i} \right) - \sum_{i=1}^n \widehat{V}_i(\underline{\theta}_i; \mathbf{x}, \mathbf{t}_i) \right\} dF(\theta) \quad (19)$$

Thus, P1 is equivalent to an alternative problem $\mathbf{P1}'$ that maximizes (19) subject to (1), (2), (3) and (18). ■

Based on Proposition 3, we can construct an optimal mechanism in which $\mathbf{x}^*$ can be independently determined and then $\mathbf{t}^*$ is subsequently determined by $\mathbf{x}^*$. Moreover, the optimal objective value of PL can be uniquely determined by $\mathbf{x}^*$ independent of $\mathbf{t}^*$.

Corollary 1 *Suppose that there are no substantive Bayesian update beliefs. Under assumptions*

1-3, suppose that

$$\begin{aligned} \mathbf{x}^* &\in \arg \max_{\mathbf{x}} \int_{\Theta} u(\mathbf{x}(\theta), \theta) + \sum_{i=1}^n (v_i(\mathbf{x}(\theta), \theta) \\ &\quad - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial v_i(\mathbf{x}(\theta), \theta_i, \theta_{-i})}{\partial \theta_i}) dF(\theta) \\ &\quad s.t. \int_{\Theta_{-i}} \frac{\partial v_i(\mathbf{x}(h, \theta_{-i}), \theta_i, \theta_{-i})}{\partial \theta_i} dF_{-i}(\theta_{-i}) \text{ is increasing in } h \text{ on } \Theta_i \\ \forall \theta &\in \Theta, \mathbf{x}(\theta) \in X. \end{aligned}$$

In addition, $\forall i \in \mathcal{N}, \theta \in \Theta$,

$$\begin{aligned} \mathbf{t}_i^*(\theta) \text{ is such that } \int_{\Theta_{-i}} \mathbf{t}_i^*(\theta) dF_{-i}(\theta_{-i}) &= \int_{\Theta_{-i}} \left[\int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}^*(s_i, \theta_{-i}), s_i, \theta_{-i})}{\partial \theta_i} ds_i - v_i(\mathbf{x}^*(\theta), \theta) \right] dF_{-i}(\theta_{-i}) + r_i, \\ \text{as long as } \mathbf{t}(\theta) &\in T, \forall \theta \in \Theta, \\ \text{where } \int_{\Theta_{-i}} \mathbf{t}_i(\theta) dF_{-i}(\theta_{-i}) &= \int_{\Theta_{-i}} \left[\int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}(s_i, \theta_{-i}), s_i, \theta_{-i})}{\partial \theta_i} ds_i - v_i(\mathbf{x}(\theta), \theta) \right] dF_{-i}(\theta_{-i}) + r_i. \end{aligned}$$

Then such a combination $(\mathbf{x}^*, \mathbf{t}^*)$ is the **optimal mechanism solving P1**.

Moreover, the optimal objective value of PL in **P1** is always equal to

$$\int_{\Theta} u(\mathbf{x}^*(\theta), \theta) + \sum_{i=1}^n (v_i(\mathbf{x}^*(\theta), \theta) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial v_i(\mathbf{x}^*(\theta), \theta_i, \theta_{-i})}{\partial \theta_i}) dF(\theta) - \sum_{i=1}^n r_i.$$

Proof. (2) and (3) give

$$\begin{aligned} r_i &\leq \widehat{V}_i(\underline{\theta}_i; \mathbf{x}, \mathbf{t}_i) = \widehat{V}_i(\theta_i; \mathbf{x}, \mathbf{t}_i) - \int_{\underline{\theta}_i}^{\theta_i} \int_{\Theta_{-i}} \frac{\partial v_i(\mathbf{x}(s_i, \theta_{-i}), s_i, \theta_{-i})}{\partial \theta_i} dF_{-i}(\theta_{-i}) ds_i \\ &= \int_{\Theta_{-i}} v_i(\mathbf{x}(\theta), \theta) + \mathbf{t}_i(\theta) dF_{-i}(\theta_{-i}) - \int_{\underline{\theta}_i}^{\theta_i} \int_{\Theta_{-i}} \frac{\partial v_i(\mathbf{x}(s_i, \theta_{-i}), s_i, \theta_{-i})}{\partial \theta_i} dF_{-i}(\theta_{-i}) ds_i \\ &= \int_{\Theta_{-i}} v_i(\mathbf{x}(\theta), \theta) + \mathbf{t}_i(\theta) dF_{-i}(\theta_{-i}) - \int_{\Theta_{-i}} \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}(s_i, \theta_{-i}), s_i, \theta_{-i})}{\partial \theta_i} ds_i dF_{-i}(\theta_{-i}) \\ &= \int_{\Theta_{-i}} \{v_i(\mathbf{x}(\theta), \theta) + \mathbf{t}_i(\theta) - \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}(s_i, \theta_{-i}), s_i, \theta_{-i})}{\partial \theta_i} ds_i\} dF_{-i}(\theta_{-i}) \end{aligned}$$

Thus, if PL can pick $\mathbf{t}_i$ such that

$$\int_{\Theta_{-i}} \mathbf{t}_i(\theta) dF_{-i}(\theta_{-i}) = \int_{\Theta_{-i}} \left[\int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}(s_i, \theta_{-i}), s_i, \theta_{-i})}{\partial \theta_i} ds_i - v_i(\mathbf{x}(\theta), \theta) \right] dF_{-i}(\theta_{-i}) + r_i \quad (20)$$

for each i as long as $\mathbf{t}(\theta) \in T$, she satisfies both (2) and (3), and she gets $\sum_{i=1}^n \widehat{V}_i(\underline{\theta}_i; \mathbf{x}, \mathbf{t}_i) = \sum_{i=1}^n r_i$, which is the best possible value for this term in (19). Therefore, we can entirely drop $\mathbf{t}$ from

P1'. Moreover, the objective function (19) in **P1'** can be simplified to

$$\int_{\Theta} u(\mathbf{x}(\theta), \theta) + \sum_{i=1}^n (v_i(\mathbf{x}(\theta), \theta) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial v_i(\mathbf{x}(\theta), \theta_i, \theta_{-i})}{\partial \theta_i}) dF(\theta) \quad (21)$$

and (1) and (18) are the only constraints left. This completes the proof. ■

4.5.2 Characterization of Optimal EPIC Mechanisms

Proposition 6 *Under assumptions 1-3, $(\mathbf{x}^*, \mathbf{t}^*)$ is the optimal mechanism solving P2 iff*

$$\begin{aligned} (\mathbf{x}^*, \mathbf{t}^*) &\in \arg \max_{\mathbf{x}, \mathbf{t}} \int_{\Theta} \{u(\mathbf{x}(\theta), \theta) \\ &\quad + \sum_{i=1}^n (v_i(\mathbf{x}(\theta), \theta) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial v_i(\mathbf{x}(\theta), \theta_i, \theta_{-i})}{\partial \theta_i})\} dF(\theta) \\ &\quad - \sum_{i=1}^n \int_{\Theta} \bar{V}_i(\theta_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) dF(\theta) \\ \text{s.t. } \forall \theta_{-i} &\in \Theta_{-i}, \frac{\partial v_i(\mathbf{x}(\theta, \theta_{-i}), \theta_i, \theta_{-i})}{\partial \theta_i} \text{ is increasing in } \theta_i \text{ on } \Theta_i, \\ \bar{V}_i(b_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) - \bar{V}_i(a_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) &= \int_{[a_i, b_i]} \frac{\partial v_i(\mathbf{x}(\theta), \theta)}{\partial \theta_i} d\theta_i, \forall a_i, b_i \in \Theta_i, \\ \bar{V}_i(\theta_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) &\geq r_i, \\ \text{and } \forall \theta &\in \Theta, \mathbf{x}(\theta) \in X, \mathbf{t}(\theta) \in T \end{aligned}$$

Proof. We can rewrite PL's objective function as

$$\begin{aligned} \hat{U}(\mathbf{x}, \mathbf{t}) &= \int_{\Theta} u(\mathbf{x}(\theta), \theta) - \sum_{i=1}^n \mathbf{t}_i(\theta) dF(\theta) \\ &= \int_{\Theta} u(\mathbf{x}(\theta), \theta) dF(\theta) + \int_{\Theta} \sum_{i=1}^n v_i(\mathbf{x}(\theta), \theta) dF(\theta) - \int_{\Theta} \sum_{i=1}^n v_i(\mathbf{x}(\theta), \theta) + \mathbf{t}_i(\theta) dF(\theta) \end{aligned}$$

By proposition 3,

$$\begin{aligned}
& \int_{\Theta} v_i(\mathbf{x}(\theta), \theta) + \mathbf{t}_i(\theta) dF(\theta) \\
&= \int_{\Theta} \bar{V}_i(\theta_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) dF(\theta) \\
&= \int_{\Theta} \bar{V}_i(\theta_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) dF(\theta) + \int_{\Theta} \int_{\theta_i}^{\theta_i} \frac{\partial \bar{V}_i(s_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i)}{\partial \theta_i} ds_i dF(\theta) \\
&= \int_{\Theta} \bar{V}_i(\theta_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) dF(\theta) + \int_{\Theta_i} \int_{\theta_{-i}} \int_{\theta_i}^{\theta_i} \frac{\partial \bar{V}_i(s_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i)}{\partial \theta_i} ds_i dF_{-i}(\theta_{-i}|\theta_i) dF_i(\theta_i) \\
&= \int_{\Theta} \bar{V}_i(\theta_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) dF(\theta) + \int_{\Theta_i} \int_{\theta_i}^{\theta_i} \left(\int_{\Theta_{-i}} \frac{\partial \bar{V}_i(s_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i)}{\partial \theta_i} dF_{-i}(\theta_{-i}|s_i) \right) ds_i f_i(\theta_i) d\theta_i \\
&= \int_{\Theta} \bar{V}_i(\theta_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) dF(\theta) + \int_{\Theta_i} \int_{s_i}^{\bar{\theta}_i} \left(\int_{\Theta_{-i}} \frac{\partial \bar{V}_i(s_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i)}{\partial \theta_i} dF_{-i}(\theta_{-i}|\theta_i) \right) f_i(\theta_i) d\theta_i ds_i \\
&= \int_{\Theta} \bar{V}_i(\theta_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) dF(\theta) + \int_{\Theta_i} (1 - F_i(s_i)) \int_{\Theta_{-i}} \frac{\partial v_i(\mathbf{x}(s_i, \theta_{-i}), s_i, \theta_{-i})}{\partial \theta_i} dF_{-i}(\theta_{-i}|s_i) ds_i \\
&= \int_{\Theta} \bar{V}_i(\theta_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) dF(\theta) + \int_{\Theta_i} \int_{\Theta_{-i}} (1 - F_i(\theta_i)) \frac{\partial v_i(\mathbf{x}(\theta_i, \theta_{-i}), \theta_i, \theta_{-i})}{\partial \theta_i} dF_{-i}(\theta_{-i}|\theta_i) d\theta_i \\
&= \int_{\Theta} \bar{V}_i(\theta_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) dF(\theta) + \int_{\Theta} (1 - F_i(\theta_i)) \frac{\partial v_i(\mathbf{x}(\theta_i, \theta_{-i}), \theta_i, \theta_{-i})}{\partial \theta_i} f_{-i}(\theta_{-i}|\theta_i) d\theta \\
&= \int_{\Theta} \bar{V}_i(\theta_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) dF(\theta) + \int_{\Theta} \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial v_i(\mathbf{x}(\theta_i, \theta_{-i}), \theta_i, \theta_{-i})}{\partial \theta_i} f(\theta) d\theta
\end{aligned}$$

So,

$$\begin{aligned}
\hat{U}(\mathbf{x}, \mathbf{t}) &= \int_{\Theta} \{u(\mathbf{x}(\theta), \theta) + \sum_{i=1}^n (v_i(\mathbf{x}(\theta), \theta) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial v_i(\mathbf{x}(\theta_i, \theta_{-i}), \theta_i, \theta_{-i})}{\partial \theta_i})\} f(\theta) d\theta \\
&\quad - \sum_{i=1}^n \int_{\Theta} \bar{V}_i(\theta_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) dF(\theta)
\end{aligned} \tag{22}$$

Thus, P1 is equivalent to an alternative problem **P1'** maximizing (22) subject to (9), (10), (11) and (18). ■

Based on Proposition 4, we can construct an optimal mechanism in which $\mathbf{x}^*$ can be independently determined and then $\mathbf{t}^*$ is subsequently determined by $\mathbf{x}^*$. Moreover, the optimal objective value of PL is uniquely determined by $\mathbf{x}^*$ independent of $\mathbf{t}^*$.

Corollary 2 *Under assumptions 1-3, if there exists*

$$\begin{aligned}
\mathbf{x}^* &\in \arg \max_{\mathbf{x}} \int_{\Theta} \{u(\mathbf{x}(\theta), \theta) \\
&\quad + \sum_{i=1}^n (v_i(\mathbf{x}(\theta), \theta) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial v_i(\mathbf{x}(\theta_i, \theta_{-i}), \theta_i, \theta_{-i})}{\partial \theta_i})\} dF(\theta) \\
s.t. \ \forall \theta_{-i} &\in \Theta_{-i}, \frac{\partial v_i(\mathbf{x}(\theta_i, \theta_{-i}), \theta_i, \theta_{-i})}{\partial \theta_i} \text{ is increasing in } h \text{ on } \Theta_i, \\
\forall \theta &\in \Theta, \mathbf{x}(\theta) \in X. \\
\text{and if } \forall i &\in \mathcal{N}, \theta \in \Theta, \\
\mathbf{t}_i^*(\theta) &= \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}^*(s_i, \theta_{-i}), s_i, \theta_{-i})}{\partial \theta_i} ds_i - v_i(\mathbf{x}^*(\theta), \theta) + r_i, \\
\text{as long as } \mathbf{t}(\theta) &\in T, \forall \theta \in \Theta, \\
\text{where } \mathbf{t}_i(\theta) &= \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}(s_i, \theta_{-i}), s_i, \theta_{-i})}{\partial \theta_i} ds_i - v_i(\mathbf{x}(\theta), \theta) + r_i,
\end{aligned}$$

then such a combination $(\mathbf{x}^*, \mathbf{t}^*)$ is the **optimal mechanism solving P2**.

Moreover, the optimal objective value of PL in **P2** is always equal to

$$\int_{\Theta} \{u(\mathbf{x}^*(\theta), \theta) + \sum_{i=1}^n (v_i(\mathbf{x}^*(\theta), \theta) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial v_i(\mathbf{x}^*(\theta), \theta_i, \theta_{-i})}{\partial \theta_i})\} dF(\theta) - \sum_{i=1}^n r_i.$$

Proof. $\forall \theta_{-i} \in \Theta_{-i}$, (15) and (16) give

$$\begin{aligned}
r_i &\leq \bar{V}_i(\underline{\theta}_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) = \bar{V}_i(\theta_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) - \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}(s_i, \theta_{-i}), s_i, \theta_{-i})}{\partial \theta_i} ds_i \\
&= v_i(\mathbf{x}(\theta), \theta) + \mathbf{t}_i(\theta) - \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}(s_i, \theta_{-i}), s_i, \theta_{-i})}{\partial \theta_i} ds_i \\
&= v_i(\mathbf{x}(\theta), \theta) + \mathbf{t}_i(\theta) - \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}(s_i, \theta_{-i}), s_i, \theta_{-i})}{\partial \theta_i} ds_i \\
&= v_i(\mathbf{x}(\theta), \theta) + \mathbf{t}_i(\theta) - \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}(s_i, \theta_{-i}), s_i, \theta_{-i})}{\partial \theta_i} ds_i
\end{aligned}$$

Thus, if PL can pick

$$\mathbf{t}_i(\theta) = \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}(s_i, \theta_{-i}), s_i, \theta_{-i})}{\partial \theta_i} ds_i - v_i(\mathbf{x}(\theta), \theta) + r_i$$

for each i as long as $\mathbf{t}(\theta) \in T$, she satisfies both (10) and (11), and she gets

$$\sum_{i=1}^n \int_{\Theta} \bar{V}_i(\underline{\theta}_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) dF(\theta) = \sum_{i=1}^n r_i,$$

which is the best possible value for this term in (22). Therefore, we can entirely drop $\mathbf{t}$ from $\mathbf{P1}'$. Moreover, the objective function (22) in $\mathbf{P1}'$ can be simplified to

$$\int_{\Theta} u(\mathbf{x}(\theta), \theta) + \sum_{i=1}^n \left(v_i(\mathbf{x}(\theta), \theta) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial v_i(\mathbf{x}(\theta), \theta_i, \theta_{-i})}{\partial \theta_i} \right) dF(\theta)$$

and (9) and (18) are the only constraints left. This completes the proof. ■

4.5.3 Welfare Analysis

Corollary 1 implies that in optimum the total expected surplus, equal to total expected benefit (because the total transfer cancels out), has been distorted by an amount equal to the total expected information rents of the agents plus the welfare loss from outside options (reservation utilities). In details, the total expected surplus is $\int_{\Theta} u(\mathbf{x}(\theta), \theta) + \sum_{i=1}^n v_i(\mathbf{x}(\theta), \theta) dF(\theta)$. The total expected information rents of the agents is $\int_{\Theta} \sum_{i=1}^n \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial v_i(\mathbf{x}(\theta), \theta_i, \theta_{-i})}{\partial \theta_i} dF(\theta)$. The welfare loss from outside options is $\sum_{i=1}^n r_i$. In this case, (21) represents the virtual surplus after distortion of information rents and the welfare loss from outside options, which represents the second best situation.

The total expected information rents reflects how much the private information (adverse selection) will impair social welfare. It is clear that this distortion will positively related to the number of informed agents, the inverse of hazard rate of θ_i , $\frac{1 - F_i(\theta_i)}{f_i(\theta_i)}$, and A_i 's marginal utility of θ_i , $\frac{\partial v_i(\mathbf{x}, \theta)}{\partial \theta_i}$. Each $\frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial v_i(\mathbf{x}(\theta), \theta_i, \theta_{-i})}{\partial \theta_i}$ can also be viewed as A_i 's marginal utility of θ_i discounted by hazard rate in overall expectation.

Corollary 2 shares the similar economic intuition as Corollary 1. The only difference is that there is a stricter constraint on the feasible assignment rules for ex post implementation than Bayesian implementation.

4.6 Discussions

4.6.1 The Role of Assumption 4

In addition, if strict increasing difference (assumption 4) holds, we can simplify the condition (4) as the monotonicity properties on assignment functions directly. It leads to a more direct characterization of the EPIC mechanisms per se.

Lemma 1 *Under Assumption 4, if $\forall b_i > a_i$ in $\Theta_i, \forall \theta_{-i} \in \Theta_{-i}$,*

$$v_i(x'', b_i, \theta_{-i}) - v_i(x', b_i, \theta_{-i}) \geq v_i(x'', a_i, \theta_{-i}) - v_i(x', a_i, \theta_{-i}),$$

then $x'' \geq x'$ in $\mathcal{X}$.

Proof. Suppose that $x'' < x'$. By Assumption 4,

$$v_i(x'', b_i, \theta_{-i}) - v_i(x', b_i, \theta_{-i}) < v_i(x'', a_i, \theta_{-i}) - v_i(x', a_i, \theta_{-i})$$

It obviously reaches a contradiction. ■

Corollary 3 *Under assumption 4, condition (9) is equivalent to (9') $\mathbf{x}(h, \theta_{-i})$ is increasing in h on Θ_i .*

Proof. " $\Leftarrow$ "Assumption 3 implies that $\frac{\partial v_i(x, \theta_i, \theta_{-i})}{\partial \theta_i}$ is increasing in x . Then the rest of the proof is straightforward.

" $\Rightarrow$ " $\forall b_i > a_i$ in $\Theta_i, \forall \theta_{-i} \in \Theta_{-i}$,

$$\int_{a_i}^{b_i} \frac{\partial v_i(\mathbf{x}(b_i, \theta_{-i}), \theta)}{\partial \theta_i} d\theta_i \geq \int_{a_i}^{b_i} \frac{\partial v_i(\mathbf{x}(a_i, \theta_{-i}), \theta)}{\partial \theta_i} d\theta_i$$

Hence,

$$v_i(\mathbf{x}(b_i, \theta_{-i}), b_i, \theta_{-i}) - v_i(\mathbf{x}(b_i, \theta_{-i}), a_i, \theta_{-i}) \geq v_i(\mathbf{x}(a_i, \theta_{-i}), b_i, \theta_{-i}) - v_i(\mathbf{x}(a_i, \theta_{-i}), a_i, \theta_{-i})$$

Furthermore,

$$v_i(\mathbf{x}(b_i, \theta_{-i}), b_i, \theta_{-i}) - v_i(\mathbf{x}(a_i, \theta_{-i}), b_i, \theta_{-i}) \geq v_i(\mathbf{x}(b_i, \theta_{-i}), a_i, \theta_{-i}) - v_i(\mathbf{x}(a_i, \theta_{-i}), a_i, \theta_{-i})$$

By Lemma 1, we can conclude that $\mathbf{x}(h, \theta_{-i})$ is increasing in h on Θ_i . ■

4.6.2 Separable Assignment Rules

Now we consider a particular situation in which the assignment rules can be separable across the agents. For each $i \in \mathcal{N}$, suppose $\mathcal{X}_i \subseteq \mathbb{R}$. Let $\mathcal{X} = \prod_{i \in \mathcal{N}} \mathcal{X}_i$. Let $\mathbf{x}(\theta) = \prod_{i \in \mathcal{N}} \mathbf{x}_i(\theta)$, where each $\mathbf{x}_i$ is a Borel measurable mapping from Θ to $\mathcal{X}_i$. The agents may *have no externalities in assignments*, that is, each $v_i(x, \theta) \equiv v_i(x_i, \theta)$. In this case, we will have a little revision on corollary 3.

Corollary 4 *If the agents have no externalities in assignments, condition (9) is equivalent to (9'') $\mathbf{x}_i(h, \theta_{-i})$ is increasing in h on Θ_i , for each i .*

(9'') is actually weaker than (9'). We also need to replace (9') with (9'') in all the results concerning EPIC mechanisms accordingly. The rest of the results aforementioned will still hold.

4.6.3 Private Valuations and Dominant Strategy Implementation

If the agents have private valuations, that is, each $v_i(x, \theta) \equiv v_i(x, \theta_i)$, ex post implementation will degenerate to dominant strategy implementation. In this case, clearly, all the results aforementioned will still hold, but will appear in accordingly refined forms.

5 Applications

5.1 Comparison between EPIC and BIC Mechanisms

The most remarkable application of our characterization results is comparison and equivalence between EPIC and BIC mechanisms.

5.1.1 Equivalence between EPIC and BIC Mechanisms

Clearly, if the types are allowed to be correlated, whenever a direct mechanism is EPIC (and IR), it must be BIC (and IR). Furthermore, by propositions 1 and 2, if a direct mechanism is EPIC (and IR), it also must be a BIC (and IR) mechanism with no substantive Bayesian update beliefs. But the converse is not true at large.

Now we consider an *interim payoff-equivalence* between mechanisms.

Definition 2 A mechanism $(\mathbf{x}, \mathbf{t})$ is said to be interim payoff-equivalent to a mechanism $(\mathbf{x}', \mathbf{t}')$ if for all $i \in \mathcal{N}$ and $\theta_i \in \Theta_i$,

$$\begin{aligned} \mathbf{x}(\theta) &= \mathbf{x}'(\theta), \\ \text{and } \int_{\Theta_{-i}} \mathbf{t}_i(\theta) dF_{-i}(\theta_{-i}|\theta_i) &= \int_{\Theta_{-i}} \mathbf{t}'_i(\theta) dF_{-i}(\theta_{-i}|\theta_i), \end{aligned}$$

Clearly, in such interim payoff equivalence, for a given mechanism $(\mathbf{x}, \mathbf{t})$, the other mechanism $(\mathbf{x}', \mathbf{t}')$ can deliver same expected payoff to each agent. Moreover, such interim payoff equivalence may bring analytical advantage. Obviously, it can preserve BIC and IR properties.

Proposition 7 Under assumptions 1-3, for any BIC (and IR) mechanism $(\mathbf{x}, \mathbf{t})$ with no substantive Bayesian update beliefs, if $\frac{\partial v_i(\mathbf{x}(h, \theta_{-i}), \theta_i, \theta_{-i})}{\partial \theta_i}$ is increasing in h on Θ_i , (and $\bar{V}_i(\underline{\theta}_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) \geq r_i$), then there exists an interim-payoff-equivalent EPIC (and IR) mechanism $(\mathbf{x}, \tilde{\mathbf{t}})$.

Proof. For any BIC mechanism $(\mathbf{x}, \mathbf{t})$, by proposition 3, $(\mathbf{x}, \mathbf{t})$ must satisfy

$$\int_{\Theta_{-i}} \mathbf{t}_i(\theta_i, \theta_{-i}) dF_{-i}(\theta_{-i}) = \int_{\Theta_{-i}} \left[\int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}(h, \theta_{-i}), h, \theta_{-i})}{\partial \theta_i} dh - v_i(\mathbf{x}(\theta_i, \theta_{-i}), \theta) + \varepsilon(\theta_{-i}) \right] dF_{-i}(\theta_{-i}),$$

where $\varepsilon(\theta_{-i}) = v_i(\mathbf{x}(\underline{\theta}_i, \theta_{-i}), \underline{\theta}_i, \theta_{-i}) + \mathbf{t}_i(\underline{\theta}_i, \theta_{-i})$.

We need to show that there exists an EPIC mechanism $(\mathbf{x}, \tilde{\mathbf{t}})$ satisfying

$$\begin{aligned} \mathbf{x}(\theta) &= \mathbf{x}'(\theta), \\ \text{and } \int_{\Theta_{-i}} \mathbf{t}_i(\theta) dF_{-i}(\theta_{-i}) &= \int_{\Theta_{-i}} \tilde{\mathbf{t}}_i(\theta) dF_{-i}(\theta_{-i}). \end{aligned}$$

First by proposition 4, any mechanism $(\mathbf{x}, \mathbf{t}')$ satisfying

$$\mathbf{t}'_i(\theta_i, \theta_{-i}) = \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}(h, \theta_{-i}), h, \theta_{-i})}{\partial \theta_i} dh - v_i(\mathbf{x}(\theta_i, \theta_{-i}), \theta) + \eta(\theta_{-i}),$$

where $\eta(\theta_{-i}) = v_i(\mathbf{x}(\underline{\theta}_i, \theta_{-i}), \underline{\theta}_i, \theta_{-i}) + \mathbf{t}'_i(\underline{\theta}_i, \theta_{-i})$, must be EPIC, since it satisfies conditions () and ().

Now define $\tilde{\mathbf{t}}_i(\theta_i, \theta_{-i}) = \mathbf{t}'_i(\theta_i, \theta_{-i}) - \eta(\theta_{-i}) + \varepsilon(\theta_{-i})$.

Firstly,

$$\begin{aligned} \int_{\Theta_{-i}} \tilde{\mathbf{t}}_i(\theta) dF_{-i}(\theta_{-i}) &= \int_{\Theta_{-i}} \left[\int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}(h, \theta_{-i}), h, \theta_{-i})}{\partial \theta_i} dh - v_i(\mathbf{x}(\theta_i, \theta_{-i}), \theta) + \varepsilon(\theta_{-i}) \right] dF_{-i}(\theta_{-i}) \\ &= \int_{\Theta_{-i}} \mathbf{t}_i(\theta) dF_{-i}(\theta_{-i}). \end{aligned}$$

Secondly, we need to show $(\mathbf{x}, \tilde{\mathbf{t}})$ is EPIC. We have already know $\frac{\partial v_i(\mathbf{x}(h, \theta_{-i}), \theta_i, \theta_{-i})}{\partial \theta_i}$ is increasing in h on Θ_i . Now it suffices to show

$$\bar{V}_i(b_i, \theta_{-i}; \mathbf{x}, \tilde{\mathbf{t}}_i) - \bar{V}_i(a_i, \theta_{-i}; \mathbf{x}, \tilde{\mathbf{t}}_i) = \int_{a_i}^{b_i} \frac{\partial v_i(\mathbf{x}(\theta), \theta)}{\partial \theta_i} d\theta_i, \forall a_i, b_i \in \Theta_i,$$

In fact, for each $i \in \mathcal{N}$, $\forall \theta_{-i} \in \Theta_{-i}$,

$$\begin{aligned} \bar{V}_i(b_i, \theta_{-i}; \mathbf{x}, \tilde{\mathbf{t}}_i) - \bar{V}_i(a_i, \theta_{-i}; \mathbf{x}, \tilde{\mathbf{t}}_i) &= v_i(\mathbf{x}(b_i, \theta_{-i}), b_i, \theta_{-i}) + \tilde{\mathbf{t}}_i(b_i, \theta_{-i}) - v_i(\mathbf{x}(a_i, \theta_{-i}), a_i, \theta_{-i}) - \tilde{\mathbf{t}}_i(a_i, \theta_{-i}) \\ &= v_i(\mathbf{x}(b_i, \theta_{-i}), b_i, \theta_{-i}) + \mathbf{t}'_i(b_i, \theta_{-i}) - v_i(\mathbf{x}(a_i, \theta_{-i}), a_i, \theta_{-i}) - \mathbf{t}'_i(a_i, \theta_{-i}) \\ &= \bar{V}_i(b_i, \theta_{-i}; \mathbf{x}, \mathbf{t}'_i) - \bar{V}_i(a_i, \theta_{-i}; \mathbf{x}, \mathbf{t}'_i) \\ &= \int_{a_i}^{b_i} \frac{\partial v_i(\mathbf{x}(\theta), \theta)}{\partial \theta_i} d\theta_i, \text{ since } (\mathbf{x}, \mathbf{t}') \text{ is EPIC.} \end{aligned}$$

Therefore $(\mathbf{x}, \tilde{\mathbf{t}})$ is EPIC.

As for IR, we know

$$\begin{aligned} \bar{V}_i(\underline{\theta}_i, \theta_{-i}; \mathbf{x}, \tilde{\mathbf{t}}_i) &= v_i(\mathbf{x}(\underline{\theta}_i, \theta_{-i}), \underline{\theta}_i, \theta_{-i}) + \tilde{\mathbf{t}}_i(\underline{\theta}_i, \theta_{-i}) \\ &= v_i(\mathbf{x}(\underline{\theta}_i, \theta_{-i}), \underline{\theta}_i, \theta_{-i}) + \mathbf{t}'_i(\underline{\theta}_i, \theta_{-i}) - [v_i(\mathbf{x}(\underline{\theta}_i, \theta_{-i}), \underline{\theta}_i, \theta_{-i}) + \mathbf{t}'_i(\underline{\theta}_i, \theta_{-i})] \\ &\quad + v_i(\mathbf{x}(\underline{\theta}_i, \theta_{-i}), \underline{\theta}_i, \theta_{-i}) + \mathbf{t}_i(\underline{\theta}_i, \theta_{-i}) \\ &= v_i(\mathbf{x}(\underline{\theta}_i, \theta_{-i}), \underline{\theta}_i, \theta_{-i}) + \mathbf{t}_i(\underline{\theta}_i, \theta_{-i}) \\ &= \bar{V}_i(\underline{\theta}_i, \theta_{-i}; \mathbf{x}, \mathbf{t}_i) \geq r_i \cdot (\text{by hypothesis}) \end{aligned}$$

Therefore $(\mathbf{x}, \tilde{\mathbf{t}})$ is also IR. ■

This proposition can be viewed as a generalization of proposition 4 in Mookherjee and Reichelstein (1992), which is only effective in the private valuation context with the one-

dimensional condensation property.

5.1.2 Comparison between Optimal EPIC and BIC Mechanisms

By comparison between corollaries 1 and 2, we can see that optimal EPIC mechanisms will be weakly dominated by optimal BIC mechanisms with no substantive Bayesian update beliefs in quasilinear environments. This indicates that the effect of optimal EPIC mechanism design even cannot exceed that of optimal BIC mechanism design with no substantive Bayesian update beliefs.

In light of Manelli and Vincent's work (2010), in the linear environment with private valuations, we can establish the equivalence of optimal dominant strategy IC (DSIC) mechanisms and optimal BIC mechanism design with no substantive Bayesian update beliefs. Consider a *linear environment*, as a reduced form of quasi-linear environment, in which each agent's payoff is linear in both assignment rule and transfer rule. In detail, $v_i(x, \theta) \equiv \theta_i x_i$, and $u(x, \theta) \equiv \sum_{i=1}^n z_i(\theta_i) x_i + c(\theta)$, where $z_i : \Theta_i \rightarrow \mathbb{R}$ and $c : \Theta \rightarrow \mathbb{R}$ are continuous functions. There are also separable assignment rules $\mathbf{x}(\theta) = \prod_{i \in \mathcal{N}} \mathbf{x}_i(\theta)$ satisfying $\forall \theta \in \Theta, \mathbf{x}(\theta) \in X = \{x : \sum_i x_i \leq 1 \text{ and } x_i \geq 0 \text{ for each } i\}$.

According to Manelli and Vincent (2010), for any BIC mechanism $(\mathbf{x}, \mathbf{t})$, there exists a dominant strategy IC mechanism $(\mathbf{x}', \mathbf{t}')$ that generates the same expected assignment rules, i.e. $\int_{\Theta_{-i}} \mathbf{x}_{-i}(\theta) dF(\theta_{-i}) = \int_{\Theta_{-i}} \mathbf{x}'_{-i}(\theta) dF(\theta_{-i})$ almost everywhere for each i .

Assumption 4 automatically applies in this linear environment. By corollary 1, if there exists $(\mathbf{x}^*, \mathbf{t}^*)$ as below,

$$\begin{aligned} \mathbf{x}^* &\in \arg \max_{\mathbf{x}'} \int_{\Theta} \sum_{i=1}^n z_i(\theta_i) \mathbf{x}_i(\theta) + c(\theta) + \sum_{i=1}^n \theta_i \mathbf{x}_i(\theta) \\ &\quad - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \mathbf{x}_i(\theta) dF(\theta) \\ &\quad \text{s.t. } \mathbf{x}_i(h, \theta_{-i}) \text{ is increasing in } h \text{ on } \Theta_i, \text{ for each } i. \\ \forall \theta &\in \Theta, \mathbf{x}(\theta) \in X. \\ \text{and } \forall i &\in \mathcal{N}, \theta \in \Theta, \\ \mathbf{t}_i^*(\theta) \text{ is such that } \int_{\Theta_{-i}} \mathbf{t}_i^*(\theta) dF_{-i}(\theta_{-i}) &= \int_{\Theta_{-i}} \left[\int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}^*(s_i, \theta_{-i}), s_i, \theta_{-i})}{\partial \theta_i} ds_i - v_i(\mathbf{x}^*(\theta), \theta) \right] dF_{-i}(\theta_{-i}) + r_i, \\ &\quad \text{as long as } \mathbf{t}(\theta) \in T, \forall \theta \in \Theta, \\ \text{where } \int_{\Theta_{-i}} \mathbf{t}_i(\theta) dF_{-i}(\theta_{-i}) &= \int_{\Theta_{-i}} \left[\int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}(s_i, \theta_{-i}), s_i, \theta_{-i})}{\partial \theta_i} ds_i - v_i(\mathbf{x}(\theta), \theta) \right] dF_{-i}(\theta_{-i}) + r_i, \end{aligned}$$

then $(\mathbf{x}^*, \mathbf{t}^*)$ is the optimal solution to **P1** with no substantive Bayesian update beliefs.

Note that

$$\begin{aligned}
& \int_{\Theta} \sum_{i=1}^n z_i(\theta_i) \mathbf{x}_i(\theta) + c(\theta) + \sum_{i=1}^n (\theta_i \mathbf{x}_i(\theta) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \mathbf{x}_i(\theta)) dF(\theta) \\
&= \sum_{i=1}^n \left\{ \int_{\Theta_i} z_i(\theta_i) \left[\int_{\Theta_{-i}} \mathbf{x}_i(\theta) dF_{-i}(\theta_{-i}) \right] dF_i(\theta_i) \right\} + \int_{\Theta} c(\theta) dF(\theta) \\
&\quad + \sum_{i=1}^n \left\{ \int_{\Theta_i} \left(\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right) \left[\int_{\Theta_{-i}} \mathbf{x}_i(\theta) dF_{-i}(\theta_{-i}) \right] dF_i(\theta_i) \right\} \\
&= \sum_{i=1}^n \left\{ \int_{\Theta_i} z_i(\theta_i) \left[\int_{\Theta_{-i}} \mathbf{x}'_i(\theta) dF_{-i}(\theta_{-i}) \right] dF_i(\theta_i) \right\} + \int_{\Theta} c(\theta) dF(\theta) \\
&\quad + \sum_{i=1}^n \left\{ \int_{\Theta_i} \left(\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right) \left[\int_{\Theta_{-i}} \mathbf{x}'_i(\theta) dF_{-i}(\theta_{-i}) \right] dF_i(\theta_i) \right\} \\
&= \int_{\Theta} z_i(\theta_i) \mathbf{x}'_i(\theta) + c(\theta) + \sum_{i=1}^n (\theta_i \mathbf{x}'_i(\theta) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \mathbf{x}'_i(\theta)) dF(\theta)
\end{aligned}$$

By Manelli and Vincent (2010), for any BIC mechanism $(\mathbf{x}, \mathbf{t})$, there exists a dominant strategy IC mechanism $(\mathbf{x}', \mathbf{t}')$ that generates the same expected assignment rules, i.e. $\int_{\Theta_{-i}} \mathbf{x}_{-i}(\theta) dF(\theta_{-i}) = \int_{\Theta_{-i}} \mathbf{x}'_{-i}(\theta) dF(\theta_{-i})$ almost everywhere for each i . Thus there must exist

$$\begin{aligned}
\mathbf{x}^{**} &\in \arg \max_{\mathbf{x}'} \int_{\Theta} \sum_{i=1}^n z_i(\theta_i) \mathbf{x}'_i(\theta) + c(\theta) + \sum_{i=1}^n \theta_i \mathbf{x}'_i(\theta) \\
&\quad - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \mathbf{x}'_i(\theta) dF(\theta) \\
&\quad \text{s.t. } \int_{\Theta_{-i}} \mathbf{x}'_i(h, \theta_{-i}) dF_{-i}(\theta_{-i}) \text{ is increasing in } h \text{ on } \Theta_i, \\
\forall \theta &\in \Theta, \mathbf{x}(\theta) \in X.
\end{aligned}$$

Let

$$\mathbf{t}_i^{**}(\theta) = \int_{\underline{\theta}_i}^{\theta_i} \frac{\partial v_i(\mathbf{x}^{**}(s_i, \theta_{-i}), s_i, \theta_{-i})}{\partial \theta_i} ds_i - v_i(\mathbf{x}^*(\theta), \theta) + r_i.$$

Then $(\mathbf{x}^{**}, \mathbf{t}^{**})$ is DSIC and solves **P2**. It will bring to PL the same objective value as $(\mathbf{x}^*, \mathbf{t}^*)$ does. The optimal objective value is equal to

$$\int_{\Theta} \sum_{i=1}^n z_i(\theta_i) \mathbf{x}_i^*(\theta) + c(\theta) + \sum_{i=1}^n \left(\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right) \mathbf{x}_i^*(\theta) dF(\theta) - \sum_{i=1}^n r_i$$

If $u(x, \theta) \equiv \sum_{i=1}^n \theta_i x_i$, both $(\mathbf{x}^*, \mathbf{t}^*)$ and $(\mathbf{x}^{**}, \mathbf{t}^{**})$ are efficient mechanisms with the same optimal objective value (normally social welfare).

This can also reflect how much efficiency will lose if we restrict attention from optimal BIC

mechanism design to optimal DSIC mechanism design when the types can allow to be correlated in such a linear environment. We can simply transfer the comparison between these two to the comparison between general optimal BIC mechanism design and optimal BIC mechanism design without substantive Bayesian update beliefs.

5.2 Comparison between Mechanisms and Menus

In Chen (2012)'s work, characterization of EPIC mechanisms can also help identify the conditions for strategic dominance between centralized mechanism design and decentralized menu design.

A **joint-based mechanism** is a pair of functions $(\mathbf{x}, \mathbf{t})$ satisfying $\mathbf{x}(\theta) \in X \subseteq \mathcal{X}$ and $\mathbf{t}(\theta) \in T \subseteq \mathbb{R}^n$. An **individual-based mechanism** is a pair of functions $(\bar{\mathbf{x}}, \bar{\mathbf{t}})$ satisfying $\bar{\mathbf{x}}(\theta) \in X \subseteq \mathcal{X}$ and $\bar{\mathbf{t}}(\theta) \in T \subseteq \mathbb{R}^n$, where $\bar{\mathbf{x}}(\theta) = (\bar{\mathbf{x}}_1(\theta_1), \dots, \bar{\mathbf{x}}_n(\theta_n))$ and $\bar{\mathbf{t}}(\theta) = (\bar{\mathbf{t}}_1(\theta_1), \dots, \bar{\mathbf{t}}_n(\theta_n))$ for each θ . Chen shows that individual-based EPIC mechanism design is strategically equivalent to ex post menu design. Thus, comparison between centralized mechanism design and decentralized menu design gets boiled down to comparison between joint-based EPIC mechanisms and individual-based EPIC mechanisms.

Assumption 5

$$V_i(x_i, s_i, t) \equiv w_i(t)x_i + s_i,$$

where $w_i : T \rightarrow \mathbb{R}$ is the payoff function from the agents' assignments. v_i is continuous. For all t , $\frac{\partial w_i(t)}{\partial t_i} > 0$. A_i has the reservation utility $r_i \in \mathbb{R}$.

According to corollary 2, we have such a characterization result for optimal joint-based mechanisms as below.

$$\begin{aligned} \mathbf{x}^* &\in \arg \max_{\mathbf{x}} \int_T \left\{ u(\mathbf{x}(\theta), \theta) \right. \\ &\quad \left. + \sum_{i=1}^n w_i(\theta) \mathbf{x}_i(\theta) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial w_i(\theta)}{\partial \theta_i} \mathbf{x}_i(\theta) \right\} dF(\theta) \\ \text{s.t. for each } \theta_{-i} &\in T_{-i}, \mathbf{x}(\theta) \text{ is increasing in } \theta_i, \\ \text{for each } \theta &\in T, \mathbf{x}(\theta) \in X, \\ \text{and for each } i &\in \mathcal{N}, \theta \in T, \\ \mathbf{t}_i^*(\theta) &= \int_{\theta_i}^{\theta_i} \frac{\partial w_i(s_i, \theta_{-i})}{\partial \theta_i} \mathbf{x}^*(s_i, \theta_{-i}) ds_i - w_i(\theta) \mathbf{x}_i^*(\theta) + r_i, \\ \text{as long as } \mathbf{t}(\theta) &\in T, \forall \theta \in \Theta, \text{ where} \\ \mathbf{t}_i(\theta) &= \int_{\theta_i}^{\theta_i} \frac{\partial w_i(\mathbf{x}(s_i, \theta_{-i}), s_i, \theta_{-i})}{\partial \theta_i} ds_i - w_i(\mathbf{x}(\theta), \theta) + r_i, \end{aligned}$$

Then $(\mathbf{x}^*, \mathbf{t}^*)$ solves the **optimal joint-based EPIC mechanism design problem**. Moreover,

the optimal objective value of PL is always equal to

$$\int_{\theta} \left\{ u(\mathbf{x}^*(\theta), \theta) + \sum_{i=1}^n \left(v_i(\theta) \mathbf{x}_i^*(\theta) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial v_i(\theta)}{\partial \theta_i} \mathbf{x}^*(\theta) \right) \right\} dF(\theta) - \sum_{i=1}^n r_i.$$

By a similar argument, we can obtain a corresponding characterization result for optimal individual-based mechanisms as below. If

$$\begin{aligned} \bar{\mathbf{x}}^* &\in \arg \max_{\bar{\mathbf{x}}} \int_{\theta} \left\{ u(\bar{\mathbf{x}}(\theta), \theta) \right. \\ &\quad \left. + \sum_{i=1}^n \left(w_i(\theta) \bar{\mathbf{x}}_i(\theta_i) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial w_i(\theta)}{\partial \theta_i} \bar{\mathbf{x}}_i(\theta) \right) \right\} dF(\theta) \\ &\quad \text{s.t. } \bar{\mathbf{x}}_i(\theta_i) \text{ is increasing in } \theta_i, \\ \text{for each } \theta &\in \theta, \bar{\mathbf{x}}(\theta) \in X, \\ \text{and for each } i &\in \mathcal{N}, \theta \in \theta, \\ \bar{\mathbf{t}}_i^*(\theta) &= \int_{\theta_i}^{\theta_i} \frac{\partial w_i(s_i, \theta_{-i})}{\partial \theta_i} \bar{\mathbf{x}}_i^*(s_i) ds_i - w_i(\theta) \bar{\mathbf{x}}_i^*(\theta_i) + r_i, \\ \text{as long as } \bar{\mathbf{t}}(\theta) &\in T, \forall \theta \in \Theta, \text{ where} \\ \bar{\mathbf{t}}_i(\theta) &= \int_{\theta_i}^{\theta_i} \frac{\partial w_i(\mathbf{x}(s_i, \theta_{-i}), s_i, \theta_{-i})}{\partial \theta_i} ds_i - w_i(\mathbf{x}(\theta), \theta) + r_i, \end{aligned}$$

Then $(\bar{\mathbf{x}}^*, \bar{\mathbf{t}}^*)$ solves the **optimal individual-based EPIC mechanism design problem**. Moreover, the optimal objective value of PL is always equal to

$$\int_{\theta} \left\{ u(\bar{\mathbf{x}}^*(\theta), \theta) + \sum_{i=1}^n \left(w_i(\theta) \bar{\mathbf{x}}_i^*(\theta) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial w_i(\theta)}{\partial \theta_i} \bar{\mathbf{x}}_i^*(\theta) \right) \right\} dF(\theta) - \sum_{i=1}^n r_i.$$

According to Chen, we have a result for strategic dominance of joint-based EPIC mechanism design as below.

Proposition 8 *In the quasilinear environment, under assumptions 1-5, suppose that X is also convex. If*

$$u(x + b, \theta) - u(x, \theta) > \sum_{i=1}^n b_i \left(\frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial w_i(\theta)}{\partial \theta_i} - w_i(\theta) \right), \quad (23)$$

for any θ, x , and $b = \prod_{i=1}^n b_i \in \mathbb{R}^n$, joint-based EPIC mechanism design strictly dominates both individual-based EPIC mechanism design and also ex post menu design.

For any type profile, if the change in PL's payoff induced by a change in x should exceed the change in the agents' total information rent net of payoffs from assignments induced by that change in x , then joint-based mechanism can make PL strictly better off, because it can lead to greater benefit for PL than the information loss incurred by the agents.

5.3 Optimal Surplus Extraction

In the auction design context, Let x_i represent the probability of bidder i 's wining the object and m_i represent the payment from the bidder i . Assume the real-valued valuation function h_i is continuous in θ . Now the designer's objective is to extract the surplus as possible as she can. Let

$$\begin{aligned} V_i(x, t, \theta) &\equiv w_i(\theta)x_i - m_i, v_i(x, \theta) \equiv w_i(\theta)x_i \\ U(x, t, \theta) &\equiv \sum_{i=1}^n m_i - \max_{i \in \mathcal{N}} w_i(\theta). \end{aligned}$$

Set $u(x, \theta) = -\max_{i \in \mathcal{N}} v_i(\theta)$ and $t_i = -m_i$. Assume that for each i , $\frac{\partial w_i(\theta)}{\partial \theta_i} \geq 0$ and $r_i = 0$. Then in ex post implementation, by our characterization results, if

$$\begin{aligned} \mathbf{x}^* &\in \arg \max_{\mathbf{x}} \int_{\Theta} \left\{ -\max_{i \in \mathcal{N}} h_i(\theta) \right. \\ &\quad \left. + \sum_{i=1}^n \left(h_i(\theta) \mathbf{x}_i^*(\theta) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial h_i(\theta_i, \theta_{-i})}{\partial \theta_i} \mathbf{x}_i^*(\theta) \right) \right\} dF(\theta) \\ \text{s.t. } \forall \theta_{-i} &\in \Theta_{-i}, \mathbf{x}_i(\theta) \text{ is increasing in } \theta_i, \\ \forall \theta &\in \Theta, \sum_{i=1}^n \mathbf{x}_i(\theta) \leq 1, \mathbf{x}_i(\theta) \geq 0, \end{aligned}$$

then $\mathbf{x}^*$ solves **P2** and uniquely determines the optimal surplus extraction. The optimal surplus extraction should be non-positive. It is equal to

$$\int_{\Theta} \left\{ -\max_{i \in \mathcal{N}} h_i(\theta) + \sum_{i=1}^n \left(h_i(\theta) \mathbf{x}_i^*(\theta) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial h_i(\theta_i, \theta_{-i})}{\partial \theta_i} \mathbf{x}_i^*(\theta) \right) \right\} dF(\theta)$$

Its best possible value would be zero. In that case, full surplus extraction will appear. It occurs only when $\mathbf{x}^*$ satisfies

$$\int_{\Theta} \max_{i \in \mathcal{N}} h_i(\theta) dF(\theta) = \int_{\Theta} \sum_{i=1}^n \left(h_i(\theta) \mathbf{x}_i^*(\theta) - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \frac{\partial h_i(\theta_i, \theta_{-i})}{\partial \theta_i} \mathbf{x}_i^*(\theta) \right) dF(\theta)$$

6 Conclusion

Our characterization results make the analysis more tractable in quasilinear environment. It addresses more interdependence features in multi-agency situations. The results covers both BIC and EPIC mechanisms. In particular, we can characterize the optimal incentive compatible mechanisms. It will help simplify or specify the analytical issues in optimal mechanism design problems.

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